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Gracefulness of Some New Class of Graphs

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Abstract: In this paper we study the gracefulness and even gracefulness of splitting graphs and fan graphs. In addition, odd-even gracefulness of tensor product of two graphs is also proved.

Keywords: Graceful labeling, Even graceful labeling, Splitting graph, Fan graph, Tensor product of graphs.

1. INTRODUCTION

Rosa[1] and Golomb[2] have extensively studied the gracefulness of graphs. J.A.Gallian[3] has given a dynamic survey of graph labeling. S.K. Vaidya and Lekha Bijukumar[4] proved that the splitting graph of $K_{1,n}$ as well as the tensor product of $K_{1,n}$ and P_2 admit odd graceful labeling.

Definition 1.1 A graph $G = (V(G), E(G))$ with p vertices and q edges is said to admit *graceful labeling* if $f : V(G) \to \{0,1,2, \dots, q\}$ such that distinct vertices receive distinct numbers and $\{|f(u) - f(v)| \,/ uv \in E(G)\} = \{1,2,3, \dots, q\}$.

Definition 1.2 A graph $G = (V(G), E(G))$ with p vertices and q edges is said to admit *even graceful labeling* if $f : V(G) \to \{0,2,4, \dots, 2q\}$ is injective and the induced function, $f^*: E(G) \to \{2,4,6, \dots, 2q\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is bijective. A graph which admits even graceful labeling is called an *even graceful graph.*

2. SPLITTING GRAPH

Definition 2.1 For any graph$G$, the *splitting graph* is obtained by adding to each vertex v, a new vertex v' so that v' is adjacent to each and every vertex that is adjacent to v in G.

Theorem 2.2 Splitting graph of a star S_n $(or\ K_{1,n})$ admits graceful labeling.

Proof: Let $v, v_1, v_2, \dots, v_n$ be the vertices of star graph $K_{1,n}$ with v as the apex vertex. The splitting graph G of $K_{1,n}$ is obtained by adding the vertices $v', v_1', v_2', \dots, v_n'$ to $K_{1,n}$. This resultant graph consists of $2(n+1)$ vertices and $3n$ edges.

We define $f : V(G) \to \{0,1,2, \dots, q\}$ by

$$f(v) = \left(\frac{q+n}{2}\right)$$
$$f(v_i) = i - 1, 1 \le i \le n$$
$$f(v') = q$$

$$f(v_i') = i + n - 1, 1 \le i \le n$$

By using the above definition of f, we have shown that the splitting graph of a star graph $K_{1,n}$ is graceful.

Illustration 2.3 Consider a star graph $K_{1,5}$. It consists of an apex vertex v and vertices v_1, v_2, v_3, v_4 and v_5. The new graph G is obtained by adding to each and every vertex v of $K_{1,5}$ a new vertex v', such that v' is adjacent to each vertex that is adjacent to v in $K_{1,5}$. The new edges are shown with dotted lines in Figure-1. The labeling is done as per the definition of f in Theorem-1. The vertex v is labeled as 10. (Here q is 15 and n is 5). The vertices v_1, v_2, v_3, v_4, v_5 are labeled as $0, 1, 2, 3,$ 4 respectively and the vertices $v_1', v_2', v_3', v_4', v_5'$ are labeled as $5, 6, 7, 8,$ 9 respectively. Finally the vertex v' is labeled as 15.

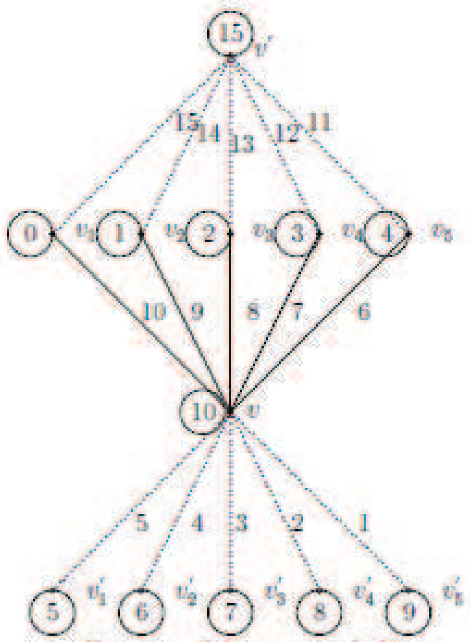

Figure 1. Splitting graph $K_{1,5}$

Illustration 2.4 Consider a star graph $K_{1,6}$. It consists of an apex vertex v and vertices $v_1, v_2, v_3, v_4, v_5, v_6$. Now the graph G is obtained by adding to each and every vertex v of $K_{1,6}$ a new vertex v', such that v' is adjacent to each vertex that is adjacent to v in $K_{1,6}$. The new edges are shown with dotted lines in Figure-2. The labeling is done as per the definition of f in Theorem-1.

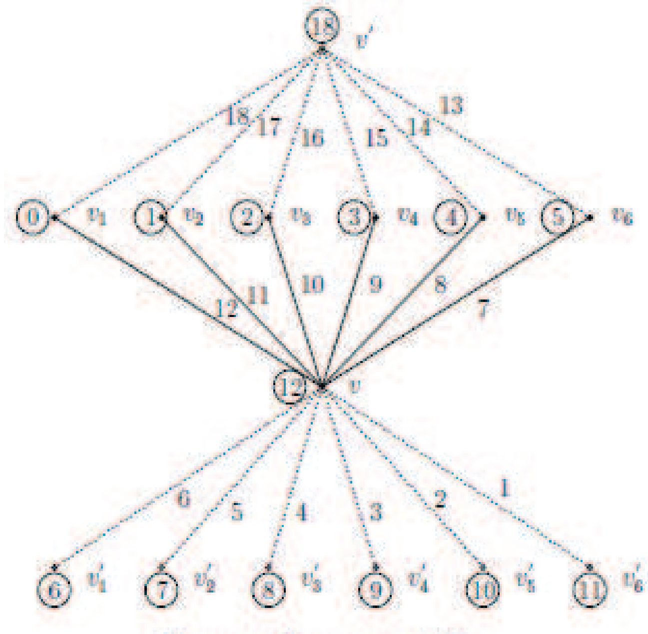

Figure 2. Splitting graph $K_{1,6}$

3. FAN GRAPH

Definition 3.1 The join $K_1 \vee P_n$ of K_1 and P_n is called a *fan graph*,F_n. The vertex of K_1 is called the core. The edges incident with the core are called spokes.

Theorem 3.2 Fan graph is graceful.

Proof: Fan graph F_n consists of $n+1$ vertices and $2n-1$ edges. This graph can be given graceful labeling by defining f as follows: $f : V(G) \to \{0,1,2,\dots,q\}$.

$$f(v_0) = 0$$
$$for\ 1 \le i \le n,$$
$$f(v_i) = \begin{cases} q-(i-1), & i \text{ is odd} \\ i-1, & i \text{ is even} \end{cases}$$

where q is the number of edges in F_n.

Illustration 3.3 The fan graph F_5 consisting of 6 vertices and 9 edges is considered for odd n. The core vertex v_0 is labeled as 0 and the other vertices v_1, v_2, v_3, v_4, v_5 are labeled as $9, 1, 7, 3, 5$ respectively. Figure-3 illustrates the labeling of this graph.

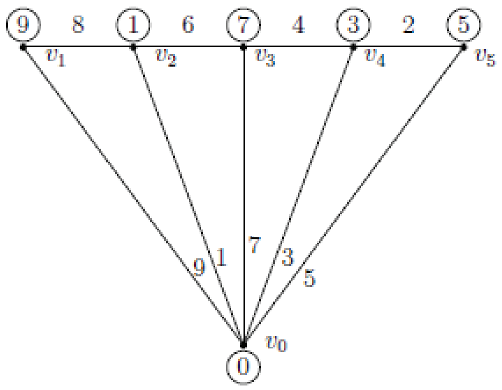

Figure 3. Fan graph F_5

Illustration 3.4 The fan graph F_6 consisting of 7 vertices and 11 edges is considered for even n and the gracefulness is shown in Figure-4.

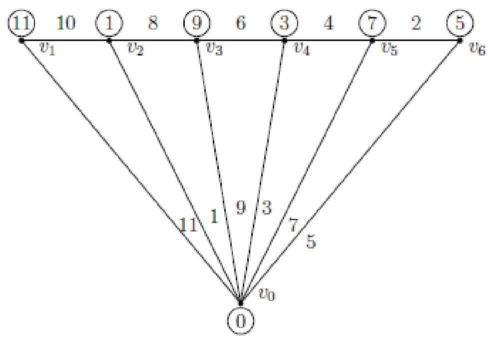

Figure 4. Fan graph F_6

4. TENSOR PRODUCT OF TWO GRAPHS

Definition 4.1 A graph $G = (V(G), E(G))$ with p vertices and q edges is said to admit *odd-even graceful labeling* if $f : V(G) \to \{0,1,2,\dots,2q\}$ is injective and the induced function, $f^* : E(G) \to \{2,4,6,\dots,2q\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is bijective. It should be noted that the vertices may be either odd or even numbers but the edges are evenly numbered. That is the reason for the name odd-even gracefulness.

Definition 4.2 The tensor product of two graphs $G_1$and G_2 is a graph denoted as $G_1 \otimes G_2$ with the vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \otimes G_2) = \{(u_1, v_1)(u_2, v_2) / u_1u_2 \in E(G_1)\ and\ v_1v_2 \in E(G_2)\}$.

Theorem 4.3 The tensor product of $K_{1,n}$ and P_2 is an odd-even graceful graph.

Proof: Let $u_1, u_2, u_3, \dots, u_n, u_{n+1}$ be the vertices of star graph $K_{1,n}$ with u_1 as the apex vertex. Let v_1 , v_2 be the vertices of the path graph P_2. Let G be the resultant graph $K_{1.n} \otimes P_2$. The number of vertices in the resultant graph is $2(n+1)$ and its number of edges is $2n$.

The vertices of G are divided into two disjoint sets T_1 and T_2.
$T_1 = \{(u_i, v_1)/i = 1,2,3, \dots, n+1\}$
$T_2 = \{(u_i, v_2)/i = 1,2,3, \dots, n+1\}$

Define the function

$f : V(G) \rightarrow \{0,1,2, \dots, 2q\}$ as
$f(u_1, v_1) = 0$
$f(u_i, v_1) = 2i - 1, 2 \le i \le n+1$
$f(u_1, v_2) = 1$
$f(u_i, v_2) = 2(n+i) - 2, 2 \le i \le n+1$

The function f defined above provides odd-even graceful labeling for tensor product of the star graph $K_{1,n}$ and the path $P_{2.}$

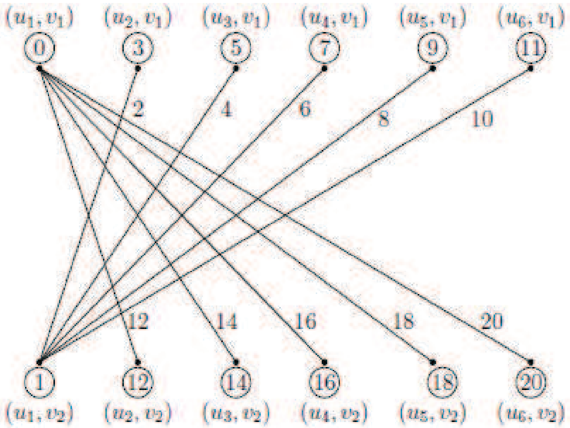

Figure 5. Tensor product of $K_{1,5}$ and P_2

Illustration 4.4 Consider a star graph $K_{1,5}$ and a path P_2. The tensor product of $K_{1,5}$ and P_2 is denoted by $K_{1.5} \otimes P_2$. The vertices are labeled as follows: The vertices $(u_1, v_1), (u_2, v_1), (u_3, v_1)$, $(u_4, v_1), (u_5, v_1)$ and (u_6, v_1) are labeled as $0, 3, 5, 7, 9$ and 11. The vertices $(u_1, v_2), (u_2, v_2), (u_3, v_2), (u_4, v_2), (u_5, v_2)$ and (u_6, v_2) are labeled as $1, 12, 14, 16, 18$ and 20. This type of labeling shows that the edges are labeled evenly. The vertices take both odd and even integer values as its labeling.

Illustration 4.5 The tensor product of $K_{1,6}$ and P_2 can also be odd-evenly labeled.

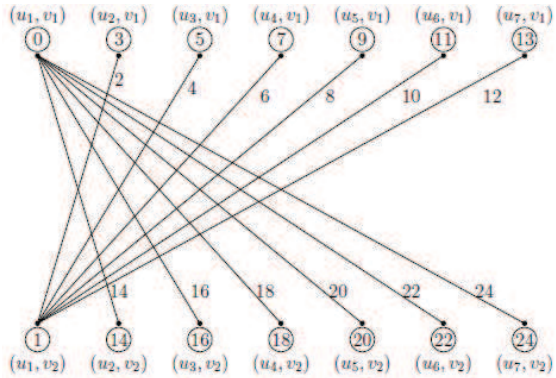

Figure 6. Tensor product of $K_{1,6}$ and P_2

5. CONCLUSION

We have proved that the splitting graphs and the fan graphs can be labeled gracefully. We have also proved that the tensor product of a complete bipartite graphs and a path can be labeled odd-even gracefully. Both splitting graph and fan graph can also be labeled even gracefully when the labeling of the vertices are doubled.

6. REFERENCES

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