ON A CLASS OF QUASI HYPERBOLIC KAC MOODY ALGEBRAS OF RANK 11

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Abstract: Kac Moody algebras, one of the modern fields of mathematical research has interesting applications to other branches of Mathematics and Mathematical Physics. The broad classification of Kac Moody algebras falls into finite , affine and indefinite types; The hyperbolic Kac Moody algebras are natural extensions of finite and affine types. A lot of research has already been undertaken in the case of finite, affine and indefinite hyperbolic types. The structure of indefinite non-hyperbolic Kac-Moody algebras remains unexplored
In this work we consider an indefinite class of non hyperbolicKac Moody algebras called the Quasi-Hyperbolic type; the complete classification of the Dynkin diagrams of rank 11 associated with the Quasi – Hyperbolic Kac Moody Algebras, which are obtained from the hyperbolic class $H^1_{(10)}$ is given; Some of the basic properties of the roots of specific classes of Quasi-hyperbolic family $QH^1_{(10)}$ are also studied.

Keywords: Kac Moody Algebras, GenerlizedCartanMatrix(GCM), Roots, Finite, Affine, indefinite, Dynkin Diagram. AMS MSC 2010 Code 17B67.

Introduction: Kac Moody algebras are named after independent papers by Victor G Kac and Robert Moody , published in 1968 [7], [12]. These Kac Moody algebras have various applications in theoretical and Mathematical Physics like conformal field theory, theory of exactly solvable models, geometry, etc. The broad classification of Kac Moody algebras falls into three main categories : finite, affine and indefinite; A natural extension of affine type is the hyperbolic class in the indefinite category; While a lot of research has already been carried out in the finite, affine and to some extent in the hyperbolic Kac Moody algebras, the in depth structure of indefinite type of Kac Moody algebras still remains unexplored.In [11] Sthanumoorthy and Uma Maheswari introduced a new class of roots called purely imaginary roots and studied about the new class of indefinite non hyperbolic Kac Moody algebras namely extended hyperbolic type; Hechun Zhang and Kaiming Zhao in [6] defined quasi finite GCM and studied the properties of imaginary roots. In [5] Feingold and Frenkel computed level 2 root multiplicities for the hyperbolic Kac Moody algebra $HA^{(1)}1$ and $HA^{(1)}_n$. In [8], [9], [10], [11] Kang has determined the structure and obtain root multiplicities for roots upto level 5 for $HA^{(1)}_1$, $HA^{(1)}_2$ and $HA^{(1)}_n$. Benkart et.al [1], [2] studied about Indefinite Kac-Moody algebras of special linear type and classical type. Strictly imaginary roots and imaginary roots whose reflections preserve root multiplicities were studied by Casperson [4] and Bennett [3]. Structure and root multiplicities for two classes of exendedhyberbolicKac-Moody algebras $EHA_1^{(1)}$ and $EHA_2^{(2)}$ for all cases were studied by Sthanumoorthy and Uma Maheswari in [17]. Root multiplicities of some classes of extended-hyperbolic Kac-Moody and extended - hyperbolic generalized Kac-Moody algebras were studied in [15] . In [19], [20] Uma Maheswari introduced the quasi affine family of Dynkin diagrams and studied about the structure of the roots and classified the Dynkin diagrams of specific classes $QAG^{(1)}_2$; Structure of indefinite quasi affine Kac Moody algebras QHG_2, $QHA_2^{(1)}$,$QAC^{(1)}_2$ and $QAD^{(2)}_3$ were studied in [20], [21], [22]. In [18] Uma Maheswari defined another class of Dynkin diagrams and associated Kac Moody algebras of quasi hyperbolic type; The classification of rank 3 Dynkin diagrams of quasi hyperbolic Kac Moody algebras are obtained in [18] and some properties of roots are studied.

In this paper, section 2 deals with the basic concepts and definitions needed for this study; section 3 gives the complete classification of Dynkin diagrams of an indefinite class of non hyperbolicKac Moody algebras called the Quasi-Hyperbolic type of rank 11, denoted as $QH_1^{(11)}$, which are obtained from the hyperbolic family of rank 10 namely, $H^1_{(10)}$; In section 4, a particular class of $QH^1_{(10)}$ is studied in depth; Some of the basic properties of the roots are also studied.

Preliminaries: For the detailed study on Kac-Moody algebras one can refer Kac[7] and Wan[23].

Definition 2.1[7]: An integer matrix $A=(a_{ij})^n_{i,j=1}$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

(i) $a_{ii} = 2 \ \forall$ i =1,2,….,n

(ii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \ \forall$ i, j = 1,2,…,n

(iii) $a_{ij} \leq 0 \ \forall$ i, j = 1,2,…,n.

Let us denote the index set of A by N = {1,…,n}. A GCM A is said to decomposable if there exist two non-empty subsets I, J ⊂ N such that I ∪ J = N and $a_{ij} = a_{ji} = 0 \ \forall$ i ∈ I and j ∈ J. If A is not decomposable, it is said to be indecomposable.

Definition 2.2[7]: A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix D = diag(q_1,…,q_n), with q_i> 0 and q_i's are rational numbers.

Definition 2.3[7]: A realization of a matrix $A = (a_{ij})_{i,j=1}^{n}$ is a triple (H, Π , Π^v) where l is the rank of A, H is a 2n - l dimensional complex vector space, $\Pi = \{\alpha_1,\ldots,\alpha_n\}$ and $\Pi^v = \{\alpha_1^v,\ldots,\alpha_n^v\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^v) = a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are called simple roots.

The root lattice generated by Π is $Q = \sum_{i=1}^{n} z\alpha_i$

The Kac-Moody algebra g(A) associated with a GCM $A = (a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i , f_i , $i = 1,2,\ldots n$ and H with the following defining relations :

$$[h, h'] = 0, \quad h, h' \in H$$
$$[e_i, f_j] = \delta_{ij}\alpha_i^v$$
$$[h, e_j] = \alpha_j(h)e_j$$
$$[h, f_j] = -\alpha_j(h)f_j \ , \ i, j \in N$$
$$(ade_i)^{1-a_{ij}} e_j = 0$$
$$(adf_i)^{1-a_{ij}} f_j = 0 \ , \forall \ i \neq j, \ i, j \in N$$

The Kac-Moody algebra g(A) has the root space decomposition

$g(A) = \oplus_{\alpha \in Q} g_\alpha(A)$ where

$g_\alpha(A) = \{x \in g(A) / [h, x] = \alpha(h)x, \textit{for all } h \in H\}$.

An element α , $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$.

Let $Q_+ = \sum_{i=1}^{n} z\alpha_i, Q$ has a partial ordering "≤" defined by $\alpha \leq \beta$ if $\beta - \alpha \in Q_+$, where α , $\beta \in Q$.

Let $\Delta = (= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Definition 2.4[7]: Define $r_i \in$ End (H^*) as $r_i(\alpha) = \alpha - \langle \alpha^v, \alpha \rangle \alpha_i$ where $\langle \alpha^v, \alpha \rangle = \alpha(\alpha_i^v)$ and i ∈ N.For each i, r_i is an invertible linear transformation of H* and r_i is called a fundamental reflection. Define the Weyl group W to be the subgroup of aut(H*) generated by {r_i, i ∈ N}. For any $\alpha \in Q$ and $\alpha = \sum_{i=1}^{n} k_i\alpha_i$, define support of α, written as suppα , by supp $\alpha = \{i \in N / k_i \neq 0\}$.

Definition 2.5[7]: The Dynkin diagram associated with the GCM A of order n is denoted by S(A) : S(A) has n vertices and vertices i and j are connected by max {|a_{ij}|,|a_{ji}|} number of lines if a_{ij}. $a_{ji} \leq 4$ and there is an arrow pointing towards i if |a_{ij}| > 1. If a_{ij}. a_{ji}> 4, i and j are connected by a bold faced edge, equipped with the ordered pair (|a_{ij}| , |a_{ji}|) of integers.

Definition 2.6[7]: A root α ∈Δ is called real, if there exists a $w \in W$ such that w(α) is a simple root, and a root which is not real is called an imaginary root. An imaginary root α is called isotropic if (α,α) = 0.

$\alpha \in \Delta_+^{im}$ is called a minimal imaginary root (MI root, for short) if α is minimal in Δ_+^{im} with respect to the partial order on H^*.

The symmetry of the root system means that we need only to prove the results for positive imaginary roots.

Definition 2.7 [7]: We define an indefinite non-hyperbolic, GCM A= $(a_{ij})_{i,j=1}^{n}$ to be of extended hyperbolic type if every proper, connected sub diagram of S(A) is of finite, affine or hyperbolic type.

Definition 2.8[18]: Let A= $(a_{ij})_{i,j=1}^{n}$, be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S (A) to be of Quasi Hyperbolic (QH) type if S (A) has a proper connected sub diagram of hyperbolic type with n-1 vertices. The GCM A is of QH type if S (A) is of QH type. We then say the Kac Moody algebra g(A) is of QH type.

Note: Every Extended hyperbolic Dynkin diagram is Quasi hyperbolic but not conversely.

Classification Of Dynkin Diagrams Of Rank 11 , Denoted By $Qh_1^{(11)}$: Let the GCM associated with the Dynkin diagram $QH_1^{(11)}$ obtained from the rank 10 hyperbolic diagram $H_1^{(10)}$ be generally given as follows where $a_{i,11} * a_{11,j}$ = 0,1,2,3,4 or >4 but not all the products are 0 simultaneously.

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1,11} \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{2,11} \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & a_{3,11} \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & a_{4,11} \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & a_{5,11} \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & a_{6,11} \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & a_{7,11} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & a_{8,11} \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & a_{9,11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & a_{10,11} \\ a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & a_{11,9} & a_{11,10} & 2 \end{pmatrix}$$

Case 1) α_{11} is connected with only one of the 10 vertices of $H_1^{(10)}$. This vertex can be selected from the 10 vertices in $10C_1$ ways. α_i and α_{11}, can be joined by any of the 9 possible edges.

α_i  α_1; can represent any of the nine possible edges: —, ⟹, ⟸, ⟺, ⇛, ⇚,  ⇚, ⇛, or a bold faced edge ━━ (a,b), where ab > 4.

Hence we will get 9 x 10 possible Dynkin diagrams in this case.

Case 2) α_{11} is connected with any two of the 10 vertices of $H_1^{(10)}$. These two vertices can be selected from the 10 vertices in $10C_2$ ways. α_i, α_{11}, and α_j, α_{11}, can be joined by any of the 9 possible edges in 9 x 9 ways. Hence there are 9 x 9 x $10C_2$ possible Dynkin diagrams.

Case 3) α_{11} is connected with any three of the 10 vertices of $H_1^{(10)}$. These three vertices can be selected from the 10 vertices in $10C_3$ ways. . Hence there are 9 x 9 x 9 x $10C_3$ possible Dynkin diagrams in this case.

Case 4) α_{11} is connected with any four of the 10 vertices of $H_1^{(10)}$. These four vertices can be selected from the 10 vertices in $10C_4$ ways. Thus, there will be 9^4 x $10C_4$ possible Dynkin diagrams in this case.

Case 5) α_{11} is connected with any five of the 10 vertices of $H_1^{(10)}$. These five vertices can be selected from the 10 vertices in $10C_5$ ways. Thus, there will be 9^5 x $10C_5$ possible Dynkin diagrams in this case.

Case 6) α_{11} is connected with any six of the 10 vertices of $H_1^{(10)}$. These six vertices can be selected from the 10 vertices in $10C_6$ ways. Thus, there will be 9^6 x $10C_6$ possible Dynkin diagrams in this case.

Case 7) α_{11} is connected with any seven of the 10 vertices of $H_1^{(10)}$. These seven vertices can be selected from the 10 vertices in $10C_7$ ways. Thus, there will be 9^7 x $10C_7$ possible Dynkin diagrams in this case.

Case 8) α_{11} is connected with any eight of the 10 vertices of $H_1^{(10)}$. These eight vertices can be selected from the 10 vertices in $10C_8$ ways. Thus, there will be 9^8 x $10C_8$ possible Dynkin diagrams in this case.

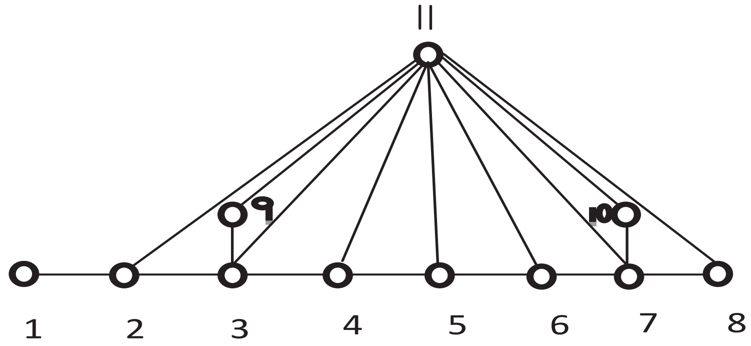
Case 9) α_{11} is connected with any nine of the 10 vertices of $H_1^{(10)}$. These nine vertices can be selected from the 10 vertices in $10C_9$ ways. Thus, there will be 9^9 x $10C_9$ possible Dynkin diagrams in this case.

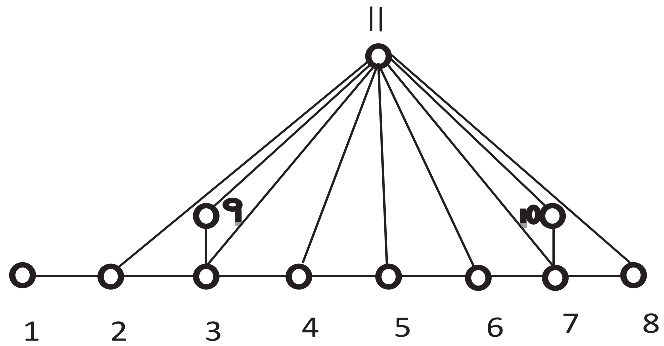
Case 10) α_{11} is connected with all the 10 vertices of $H_1^{(10)}$. Thus, there will be 9^{10} possibleDynkin diagrams in this case.

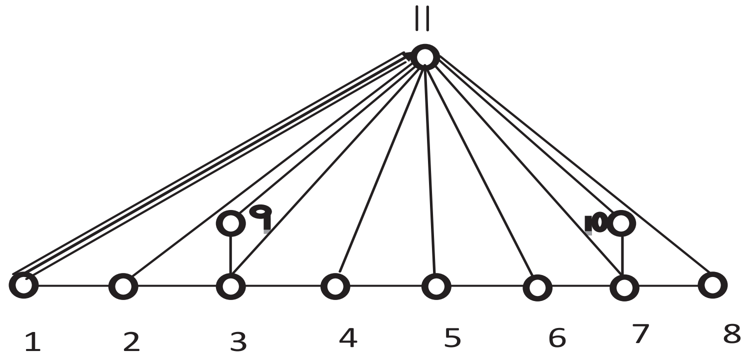
Thus, there are 9 x $10C_1$ + 9^2 x $10C_2$ + 9^3 x $10C_3$ + 9^4 x $10C_4$+ 9^5 x $10C_5$ + 9^6 x $10C_6$ + 9^7 x $10C_7$ + 9^8 x $10C_8$ + 9^9 x $10C_9$ + 9^{10}Dynkin diagrams associated with the Quasi hyperbolic type $QH_1^{(11)}$. We have proved the following theorem :

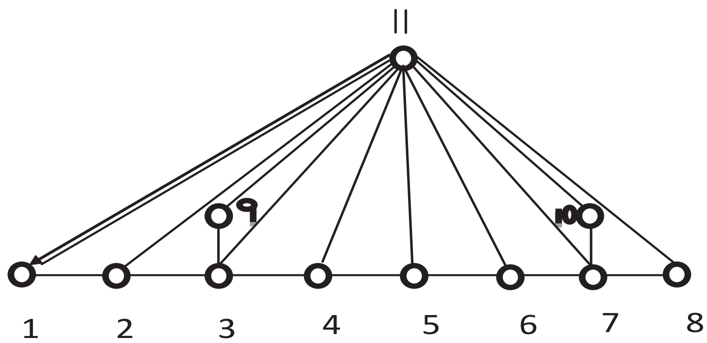
Theorem 3.1 (Classification Theorem) : Let $QH_1^{(11)}$ be the family of Quasi hyperbolic Kac Moody algebras, obtained from the rank 10 hyperbolic family $H_1^{(10)}$. Then the number of non-isomorphic connectedDynkin diagrams associated with $QH_1^{(11)}$ is $\sum_{i=1}^{10} (9^i\, 10C_i)$.

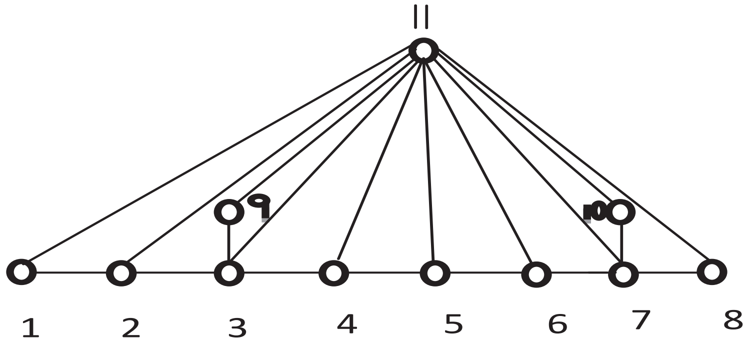
Examples of Dynkin diagrams belonging to the family $QH_1^{(11)}$

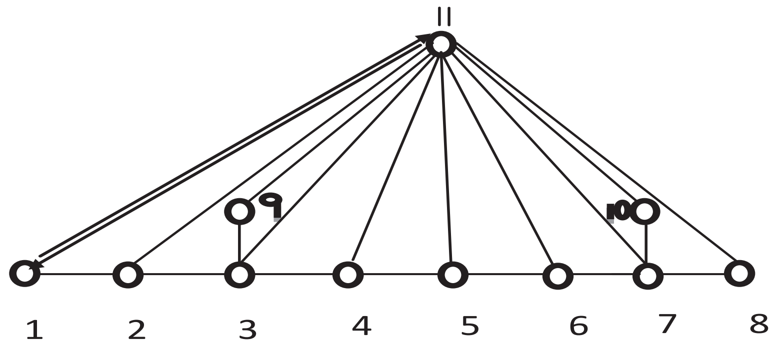


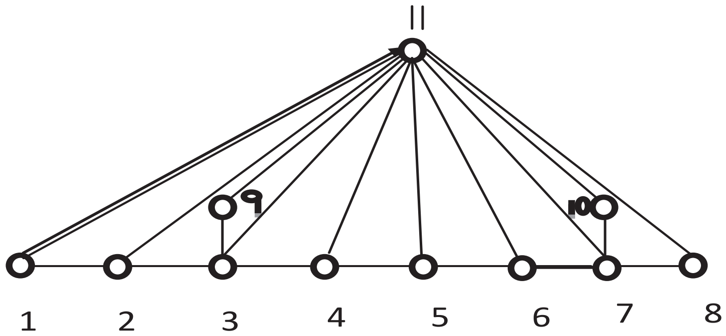








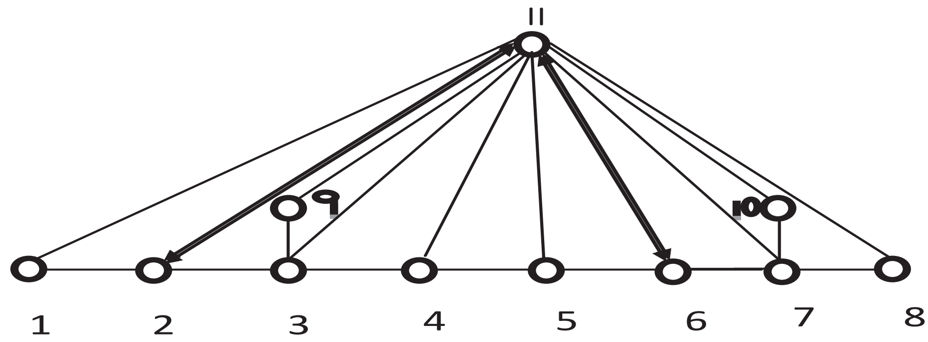




A Special Class Of Quasi Hyperbolic Kac-Moody Algebra $Qh_1^{(11)}$: In this section let us study about a particular class of quasi hyperbolic Kac Moody algebra in the family $\mathbf{QH_1^{(11)}}$. Let $\mathbf{QH_1^{(11)}}$ denote the quasi hyperbolic Kac-Moody algebra obtained from $QH_{10}^{(1)}$ whose associated GCM A= $(a_{ij})_{i,j=1}^{n}$, is given by

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ -1 & -2 & -1 & -1 & -1 & -2 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

The corresponding Dynkin diagram is



The GCM is symmetric. Therefore, bilinear form $<\alpha_i,\alpha_j>= a_{ij}$and $(\alpha_i,\alpha_j)=2$ for all i. Hence all simple roots are of equal length.Let us now tabulate the fundamental reflections r_i ,i = 1,...,11 of length 1 for this quasi hyperbolic Kac Moody algebra in Table I.

All simple roots are real roots of height 1. Let us consider the real roots of height 2: $\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$, $\alpha_3+\alpha_4$, $\alpha_4+\alpha_5$, $\alpha_5+\alpha_6$, $\alpha_6+\alpha_7$, $\alpha_7+\alpha_8$, $\alpha_3+\alpha_9$, $\alpha_7+\alpha_{10}$, $\alpha_1+\alpha_{11}$, $\alpha_3+\alpha_{11}$, $\alpha_4+\alpha_{11}$, $\alpha_5+\alpha_{11}$, $\alpha_7+\alpha_{11}$, $\alpha_8+\alpha_{11}$, $\alpha_9+\alpha_{11}$, $\alpha_{10}+\alpha_{11}$ are real roots since the length of these roots are positive;

Imaginary roots: $\alpha_6+\alpha_{11}$ and $\alpha_2+\alpha_{11}$ are imaginary roots of height 2. Among the roots of height 3, $\alpha_2+2\alpha_{11}$, $2\alpha_2+\alpha_{11}$, $\alpha_6+2\alpha_{11}$, $2\alpha_6+\alpha_{11}$are imaginary roots since the lengths of these roots are non positive and in fact, they are also minimal imaginary roots. Note that there are infinite number of imaginary roots.

Isotropic roots: Since $(\alpha_2+\alpha_{11}, \alpha_2+\alpha_{11}) = 0$ and $(\alpha_6+\alpha_{11}, \alpha_6+\alpha_{11}) = 0$, $\alpha_2+\alpha_{11}$ and $\alpha_6+\alpha_{11}$ are isotropic roots. Note that there re infinite number of isotropic roots and we have listed the isotropic roots of minimal height.

Table I.

$r_i(\alpha_j)$	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}
α_1	$-\alpha_1$	$\alpha_1+\alpha_2$	α_1	α_1	α_1	α_1	α_1	α_1	α_1	α_1	$\alpha_1+\alpha_{11}$
α_2	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_2+\alpha_3$	α_2	α_2	α_2	α_2	α_2	α_2	α_2	$\alpha_2+2\alpha_{11}$
α_3	α_3	$\alpha_2+\alpha_3$	$-\alpha_3$	$\alpha_3+\alpha_4$	α_3	α_3	α_3	α_3	α_3	$\alpha_3+\alpha_9$	$\alpha_{3+}+\alpha_{11}$
α_4	α_4	α_4	$\alpha_3+\alpha_4$	$-\alpha_4$	$\alpha_4+\alpha_5$	α_4	α_4	α_4	α_4	α_4	$\alpha_4+\alpha_{11}$
α_5	α_5	α_5	α_5	$\alpha_4+\alpha_5$	$-\alpha_5$	$\alpha_5+\alpha_6$	α_5	α_5	α_5	α_5	$\alpha_5+\alpha_{11}$
α_6	α_6	α_6	α_6	α_6	$\alpha_5+\alpha_6$	$-\alpha_6$	$\alpha_6+\alpha_7$	α_6	α_6	α_6	$\alpha_6+2\alpha_{11}$
α_7	α_7	α_7	α_7	α_7	α_7	$\alpha_6+\alpha_7$	$-\alpha_7$	$\alpha_7+\alpha_8$	α_7	$\alpha_7+\alpha_{10}$	$\alpha_7+\alpha_{11}$
α_8	α_8	α_8	α_8	α_8	α_8	α_8	$\alpha_7+\alpha_8$	$-\alpha_8$	α_8	α_8	$\alpha_8+\alpha_{11}$
α_9	α_9	α_9	$\alpha_3+\alpha_9$	α_9	α_9	α_9	α_9	α_9	$-\alpha_9$	α_9	$\alpha_9+\alpha_{11}$
α_{10}	α_{10}	α_{10}	α_{10}	α_{10}	α_{10}	α_{10}	$\alpha_7+\alpha_{10}$	α_{10}	α_{10}	$-\alpha_{10}$	$\alpha_{10}+\alpha_{11}$
α_{11}	$\alpha_1+\alpha_{11}$	$2\alpha_2+\alpha_{11}$	$\alpha_3+\alpha_{11}$	$\alpha_4+\alpha_{11}$	$\alpha_5+\alpha_{11}$	$2\alpha_6+\alpha_{11}$	$\alpha_7+\alpha_{11}$	$\alpha_8+\alpha_{11}$	$\alpha_9+\alpha_{11}$	$\alpha_{10}+\alpha_{11}$	$-\alpha_{11}$

Conclusion: The classification of Dynkin diagrams of a particular class in $QH_1^{(11)}$ has been obtained; Some properties of roots have been discussed; There is further scope for understanding the behavior of roots and the structure of the associated Quasi hyperbolic Kac Moody algebra using the homological techniques and spectral sequences theory.

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References:

1. Benkart, G.M., Kang, S.J., and Misra, K.C., "Indefinite Kac-Moody Algebras of Special Linear Type", Pacific Journal of Mathematics, Vol.170, No.2, (1995), pp. 379-404.
2. Benkart, G.M., Kang, S.J., anandMisra, K.C., "Indefinite Kac-Moody Algebras of classical type", Adv. Math., Vol. 105, (1994), pp. 76-110.
3. Bennett, C., "Imaginary roots of Kac Moody algebra whose reflections preserve root multiplicities", J.Algebra,Vol. 158, (1993), pp. 244-267.
4. Casperson, D., "Strictly imaginary roots of Kac Moody algebras", J. Algebra, Vol. 168, (1994), pp. 90-122.
5. Feingold, A.J., and Frenkel, I.B., "A hyperbolic Kac-Moody algebra and the theory of siegal modular forms of genus 2", Math. Ann., Vol. 263, (1983), pp. 87-144
6. Hechun Zhang, Kaiming Zhao, "Root Systems of a class of Kac Moody Algebras", Algebra Colloq. 1, no.4, (1994), 359-368.
7. Kac, V.G. (1990), "Infinite Dimensional Lie Algebra", 3rd ed. Cambridge: Cambridge University Press. (1990)
8. S.J. Kang, "Kac-Moody Lie algebras, spectral sequences, and the Witt formula", Trans. Amer. Math Soc., 339 (1993), 463-495.
9. S.J. Kang, "Root multiplicities of the hyperbolic Kac-Moody algebra $HA^{(1)}_1$", J. Algebra, 160, (1993), 492-593.
10. Kang, S.J., "Root multiplicities of the hyperbolic Kac-Moody Lie algebra HAn(1) ," J. Algebra, Vol. 170, (1994), pp. 277-299.
11. S.J. Kang, "On the hyperbolic Kac-Moody Lie algebra HA(1)1" , Trans. Amer.Math. Soc., 341, (1994), 623-638.
12. R.V. Moody, A new class of Lie algebras, J. Algebra, 10, (1968), 211-230.
13. Sthanumoorthy, N., and Uma Maheswari, A., "Purely imaginary roots of Kac-Moody algebras", Comm. Algebra, Vol. 24, No. 2, (1996b), pp. 677-693.
14. Sthanumoorthy, N., and Uma Maheswari, A., "Root multiplicities of extended hyperbolic Kac-Moody algebras", Comm. Algebra, Vol. 24, No. 14, (1996a), pp. 4495-4512.
15. Sthanumoorthy, N., Lilly, P.L., and Uma Maheswari, A., "Root multiplicities of some classes of extended-hyperbolic Kac-Moody and extended - hyperbolic generalized Kac-Moody algebras", Contemporary Mathematics, AMS, Vol. 343, (2004), pp. 315-347.
16. Sthanumoorthy, N., Uma Maheswari, A., and Lilly, P.L., "Extended – Hyperbolic Kac-Moody $EHA_2^{(2)}$ Algebras structure and Root Multiplicities", Comm. Algebra, Vol. 32, No. 6, (2004), pp. 2457-2476.
17. Sthanumoorthy, N., and Uma Maheswari, A., "Structure And Root Multiplicities for Two classes of Extended HyberbolicKac-Moody Algebras $EHA_1^{(1)}$ and EHA2(2) for all cases", Comm. Algebra, Vol. 40, (2012), pp. 632-665.
18. Uma Maheswari, A., "Imaginary Roots and Dynkin Diagrams of Quasi HyberbolicKac-Moody Algebras of rank 3", IJMCAR,, Vol. 4, Issue 2, (2014), pp. 19-28.
19. Uma Maheswari. A, Quasi-Affine Kac-Moody Algebras: Classification of Dynkin Diagrams $QAG^{(1)}_2$, International Journal of Mathematical Sciences, Vol.34, Issue.2, (2014) 1590-1605.
20. UmaMaheswari. A, Structure of Indefinite Quasi-Affine Kac-Moody algebras $QAD_3^{(2)}$, International Journal of Pure and Applied Mathematics, Vol 102, (2015), No.1 57-72.
21. Uma Maheswari, A., and Krishnaveni, S., "A Study on the Structure of a class of Indefinite Non-Hyperbolic Kac-Moody Algebras QHG_2", IJMCAR, Vol. 4, Issue 4, (2014), 97-110.
22. Uma Maheswari, A., and Krishnaveni, S., "On the Structure of Indefinite Quasi-Hyperbolic Kac-Moody Algebras $QHA_2^{(1)}$ ", Int.J. of Algebra, , 8, No.11 , (2014), 517 – 528.
23. Wan Zhe-Xian, "Introduction to Kac-Moody Algebra", Singapore: World Scientific Publishing Co.Pvt.Ltd, (1991).

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