# A STUDY ON GENERATING FUNCTION INVOLVING LAGUERRE POLYNOMIALS OF TWO VARIABLES

DHOUNSI, K.S., YASMEEN

**Abstract :** In this paper, we obtain linear, bilinear and bilateral generating function for associated Laguerre polynomials of two variables $\mathcal{L}_n^{(\alpha)}(x,y)$ which is closely related to generalized Laguerre polynomials of Dattoli et. al. These result provide useful extensions of the well known results of Laguerre polynomials.

**Key Words**: bilinear, bilateral, Dattoli et. al., Generating functions, Linear, Laguerre polynomials, Series rearrangement techniques.

Introduction: Two variables one index Laguerre polynomials have been given by Dattoli et. al. [2], [3], [4] and also by Pathan, Khan and Yasmin [5].

Two variable one index Laguerre polynomials defined as

$$\mathcal{L}_n(x,y) = n!\sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(n-r)!(r!)^2}, \qquad \ldots(1.1)$$

for y = 0

$$\mathcal{L}_n(x,0) = \frac{(-1)^n x^n}{n!} \qquad \ldots(1.2)$$

and are specified by the generating functions

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x,y)\, t^n = \frac{1}{1-yt}\exp\left(-\frac{xt}{1-yt}\right); |yt|<1 \qquad \ldots(1.3)$$

$\mathcal{L}_n(x,y)$ are linked to the ordinary Laguerre polynomials $L_n(x)$ by

$$\mathcal{L}_n(x,1) = L_n(x) \qquad \ldots(1.4)$$

$$\mathcal{L}_n(x,y) = y^n L_n\left(\frac{x}{y}\right), \qquad \ldots(1.5)$$

Now by [1; p.211]

$$(1-t-\nu)^{-1-\alpha} = (1-t)^{-1-\alpha}\left(1-\frac{\nu}{1-t}\right)^{-1-\alpha} \qquad \ldots(1.6)$$

$$\exp\left(\frac{-x(\nu+t)}{1-t-\nu}\right) = \exp\left(\frac{-xt}{1-t}\right)\exp\left(\frac{\left(\frac{-x}{1-t}\right)\left(\frac{\nu}{1-t}\right)}{1-\frac{\nu}{1-t}}\right) \qquad \ldots(1.7)$$

A generalization of (1.1) provided by the following definition of the generalized associated Laguerre polynomials.

$$\mathcal{L}_n^{(\alpha)}(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r (1+\alpha)_n x^r y^{n-r}}{(n-r)!r!(1+\alpha)_r}, \qquad \ldots(1.8)$$

and generating function defined by

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x,y)\, t^n = (1-yt)^{-1-\alpha}\exp\left(\frac{-xt}{1-yt}\right) \qquad \ldots(1.9)$$

So that for α = 0 equation (1.8) reduces to (1.1) and (1.9) reduces to (1.3). In this paper we shall give some basic relations and properties involving the generalized associated Laguerre polynomials $\mathcal{L}_n^{(\alpha)}(x,y)$ and then take up linear, bilinear and bilateral generating function for $\mathcal{L}_n^{(\alpha)}(x,y)$. Some new and known formulae for $\mathcal{L}_n(x,y)$ [2], [3], [4] and $L_n(x)$, [1] are derived as special cases.

**Linear Generating Function**

**Theorem**: If α is a positive integer and | yt | < 1, then

$$\sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!}\mathcal{L}_{n+k}^{(\alpha)}(x,y)\, t^n$$

$$= (1-yt)^{-(1+\alpha+k)}\exp\left(\frac{-xt}{1-yt}\right)\mathcal{L}_k^{(\alpha)}\left(\frac{x}{1-yt}, y\right) \qquad \ldots(2.1)$$

**Proof**: Consider the double series

$$S = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!}\mathcal{L}_{n+k}^{(\alpha)}(x,y)\, t^n \nu^k \qquad \ldots(2.2)$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+k)!\,y^{n+k}}{n!\,k!}L_{n+k}^{(\alpha)}\left(\frac{x}{y}\right)t^n\,\nu^k$$

Now using series rearrangement techniques and equations (1.6) and (1.7), we get

$$S=(1-yt)^{-(1+\alpha)}\left(\frac{1-y\nu}{1-yt}\right)^{-(1+\alpha)}\exp\left(\frac{-\frac{x}{y}yt}{1-yt}\right)\exp\left(\frac{\left(\frac{-x/y}{1-yt}\right)\left(\frac{y\nu}{1-yt}\right)}{1-\frac{y\nu}{1-yt}}\right)$$

$$=(1-yt)^{-(1+\alpha)}\exp\left(\frac{-xt}{1-yt}\right)\sum_{k=0}^{\infty}L_k^{(\alpha)}\left(\frac{x}{y(1-yt)}\right)\left(\frac{\nu y}{1-yt}\right)^k$$

...(2.3)

Equating (2.2) and (2.3) and comparing the coefficient of $\nu^k$, we get required result (2.1).

**Bilinear Generating Functions**

**Theorem**: If | yt ν| < 1 and α be a non negative integer, then

$$(1-yt\nu)^{-1-\alpha}\exp\left(\frac{-\nu(xt+yz)}{1-yt\nu}\right){}_0F_1\left[\begin{matrix}-;\\1+\alpha;\end{matrix}\frac{xz\nu}{(1-yt\nu)^2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n}\mathcal{L}_n^{(\alpha)}(x,y)\mathcal{L}_n^{(\alpha)}(z,t)\,\nu^n$$

...(3.1)

**Proof**: Taking right hand side

$$S=\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n}\mathcal{L}_n^{(\alpha)}(x,y)\mathcal{L}_n^{(\alpha)}(z,t)\,\nu^n$$

...(3.2)

Now using (1.5), [1;p.201(3)], series rearrangement techniques and equation (2.1), we get

$$S=\sum_{k=0}^{\infty}\frac{(1-yt\nu)^{-1-\alpha}}{(1+\alpha)_k}\exp\left(\frac{-xt\nu}{1-yt\nu}\right)L_k^{(\alpha)}\left(\frac{x}{y(1-yt\nu)}\right)\left(\frac{-yz\nu}{1-yt\nu}\right)^k$$

Again using [1; p. 206(1)], we get

$$S=(1-yt\nu)^{-1-\alpha}\exp\left(\frac{-xt\nu}{1-yt\nu}\right)\exp\left(\frac{-yz\nu}{1-yt\nu}\right){}_0F_1\left[\begin{matrix}-;\\1+\alpha;\end{matrix}\frac{\frac{x}{y}(yz\nu)}{(1-yt\nu)^2}\right]$$

$$=(1-yt\nu)^{-1-\alpha}\exp\left(\frac{-(xt+yz)\nu}{1-yt\nu}\right){}_0F_1\left[\begin{matrix}-;\\1+\alpha;\end{matrix}\frac{xz\nu}{(1-yt\nu)^2}\right],$$

where $|yt\nu| < 1$

...(3.3)

Equating (3.2) and (3.3), we get required result (3.1).

**Bilateral Generating Function**

**Theorem**: If α be a positive integer and | yt | < 1, then

$$(1-yt)^{c-1-\alpha}(1-yt-yzt)^{-c}\exp\left(\frac{-xt}{1-yt}\right)$$

$$\times\,{}_1F_1\left[\begin{matrix}c;\\1+\alpha;\end{matrix}\frac{xzt}{(1-yt)\,(1-yt+yzt)}\right]$$

$$=\sum_{n=0}^{\infty}{}_2F_1\left[\begin{matrix}-n,c;\\1+\alpha;\end{matrix}z\right]\mathcal{L}_n^{(\alpha)}(x,y)\;t^n$$

...(4.1)

**Proof**: By linear generating relation, we have

$$(1-yt)^{-c}\,{}_1F_1\left[\begin{matrix}c;\\1+\alpha;\end{matrix}\frac{-xt}{1-yt}\right]=\sum_{k=0}^{\infty}\frac{(c)_k\mathcal{L}_k^{(\alpha)}(x,y)t^k}{(1+\alpha)_k}$$

...(4.2)

Now replace x by $x(1-yt)^{-1}$, t by $-tz(1-yt)^{-1}$ and multiply both side by $(1-yt)^{-1}\exp\left(\frac{-xt}{1-yt}\right)$ then equation (4.2) reduces to

$$(1-yt)^{c-1}(1-yt+zyt)^{-c}\exp\left(\frac{-xt}{1-yt}\right)\times$$

$${}_1F_1\left[\begin{matrix}c;\\1+\alpha;\end{matrix}\frac{xzt}{(1-yt)(1-yt+yzt)}\right]$$

$$=\sum_{k=0}^{\infty}\frac{(c)_k}{(1+\alpha)_k}(-zt)^k(1-yt)^{-1-k}\times\exp\left(\frac{-xt}{1-yt}\right)\mathcal{L}_k^{(\alpha)}\left(\frac{x}{1-yt},y\right)$$

Now, using equation (2.1), we get

$$(1-yt)^{c-1}(1-yt+zyt)^{-c}\exp\left(\frac{-xt}{1-yt}\right)\times{}_1F_1\left[\begin{matrix}c;\\1+\alpha;\end{matrix}\frac{xzt}{(1-yt)(1-yt+yzt)}\right]$$

$$=\sum_{n=0}^{\infty}{}_2F_1\left[\begin{matrix}-n,c;\\1+\alpha;\end{matrix}z\right](1-yt)^{\alpha}\mathcal{L}_n^{(\alpha)}(x,y)\;t^n,$$

$$(1-yt)^{c-1-\alpha}(1-yt+zyt)^{-c}\exp\left(\frac{-xt}{1-yt}\right)\times$$

$${}_1F_1\left[\begin{matrix}c;\\1+\alpha;\end{matrix}\frac{xzt}{(1-yt)(1-yt+yzt)}\right]$$

$$=\sum_{n=0}^{\infty}{}_2F_1\left[\begin{matrix}-n,c;\\1+\alpha;\end{matrix}z\right]\mathcal{L}_n^{(\alpha)}(x,y)\,t^n$$

This is a required result (4.1).

**Special Cases**

I. For y = 1, then (2.1) reduces to

$$\sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!} L_{n+k}^{(\alpha)}(x)\, t^n = (1-t)^{-(1+\alpha+k)} \exp\left(\frac{-xt}{1-t}\right) L_k^{(\alpha)}\left(\frac{x}{1-t}\right) \quad ...(5.1)$$

which is a known result [1; p.211 (9)].

For $\alpha = 0$ and y = 1 then (2.1) reduces to [1; p.215 (25)]

II. For y = t = 1 and replace z by y then (3.1) reduces to

$$(1-\nu)^{-1-\alpha} \exp\left(\frac{-(x+y)\nu}{1-\nu}\right) {}_0F_1\left[\begin{matrix} -; & \dfrac{xy\nu}{(1-\nu)^2} \\ 1+\alpha; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{n!\, L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y)\, \nu^n}{(1+\alpha)_n} \quad ...(5.2)$$

which is a known result [1; p.212 (Th-69)].

For $\alpha = 0$, $\nu = t$, then (5.2) reduces to

$$(1-t)^{-1} \exp\left(\frac{-(x+y)\nu}{1-\nu}\right) {}_0F_1\left[\begin{matrix} -; & \dfrac{xyt}{(1-t)^2} \\ 1; & \end{matrix}\right] = \sum_{n=0}^{\infty} L_n(x) L_n(y)\, t^n$$

which is a known result [1; p.215 (26)].

III. For y = 1 and replace z by y in (4.1), then we get

$$(1-t)^{c-1-\alpha}(1-t+yt)^{-c} \exp\left(\frac{-xt}{1-t}\right) {}_1F_1\left[\begin{matrix} c; & \dfrac{xyt}{(1-t)(1-t+yt)} \\ 1+\alpha; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, c; & z \\ 1+\alpha; & \end{matrix}\right] L_n^{(\alpha)}(x)\, t^n \quad ...(5.3)$$

This is a known result [1; p.213].

For $\alpha = 0$,then (5.3) reduces to [1; p.215(28)].

**References :**


1. E.D. Rainville; Special Functions.MacMillan, New York; Reprinted by Chelsea Pub. Co., Bronx, New York, 1971.
2. G. Dattoli, and A. Torre; Operational methods and two variable Laguerre polynomials, Acc. Sc. Torino-Atti Sc. Fis., 132 (1998), 1-7.
3. G. Dattoli, S. Lorenzutta, and D. Sacchetti; A note on operational rules for Hermite and Laguerre polynomials, operational rules and Special polynomials, Internet. J. Math. Stat. Sci., Vol.9,No.2 (2000), 227-238.
4. G. Dattoli and A. Torre; Acc. Sc. di Torino-Atti. Sc. Fis. 132, 1(1998), Operational methods for associated Laguerre polynomials have also been youched in A. Wunsche, J. Phys. A32, 3179.
5. M.A. Pathan, Khan Subuhi and Ghazala Yasmin; Two variable modified Laguerre polynomials and their generating relations, South East Asian Jour. Math. and Math. Sciences, 1(2002), 19-28.



Dhounsi, K.S., - Department of Mathematics,
Maharshi Dayanand College, Sri Ganganagar – 335001, Rajasthan, India
e-mail:dhounsiksingh@gmail.com
Yasmeen - Department of Mathematics,
Government College, Kota-324010, Rajasthan,India
e-mail:yasmeen_kth@yahoo.com