A PERSPECTIVE ON $\pi\beta -$ NORMAL TOPOLOGICAL SPACES

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Abstract: The framing of this paper bears the main aim to introduce and study a weaker version of β-normality called $\pi\beta$- normality, which surely lies between β- normality and almost β- normality. It contains the fact that $\pi\beta$- normality is a topological property as well as hereditary property with respect to regularly closed subspaces. The characterization & preservation theorems in the context are presented which strengthen the evidence of the existence of such spaces . In fact, there are many $\pi\beta$- normal spaces which are not β- normal.

This paper also includes β- normality in terms of disjoint dense subsets and some basic properties. The relationships among πs- normal spaces,πp- normal spaces &$\pi\beta$- normal spaces are, here, investigated.

Last but not the list , the purpose of introducing this paper is to continue the study of the class of normal spaces, namely $\pi\beta$- normal spaces, which is a generalization of the class of πp- normal spaces &πs- normal spaces.

The effort of coining this paper is nothing but a humble dedication to the eminent mathematician Professor M.E. Abd. El Monsef who breathed his last breathing on 13th August, 2014.

Introduction & Preliminary:D.Andrijevic introduced a new class of generalized open sets in a topological space, the so called β-open sets (i.e. semi-open sets) [1]. The class of semi-pre-open sets contains all semi-open sets and pre-open sets. Professor M.E.Abd El- Monsef et al projected the fundamental properties of β-open sets & β-open continuous mappings[2] along with the study of β-closure and β-interior operators[3]. We , however, know that a set in a topological space is said to be regular open set or open domain[4] if it is the interior of its closure. And the finite union of regular open sets is said to be π-open [5] with the help of these two notions of β-open set & π-open set, the concept of a πβ-normal topological space is, here, introduced. Obviously, πβ-normality lies in between β-normality & almost β-normality and it is a weaker version of β-normality.

In the present paper, spaces(X,T) and (Y,σ) always mean topological spaces which are not assumed to satisfy any separation axioms are assumed unless explicitly mentioned.

Any other notion and symbol, not defined in this paper, may be found in the appropriate reference.

Definition (1.4) [14,15,16,17,18]:

(a) A space (X,T)is said to be pre-normal or *p*-normal(resp.s-normal,β-normal) if for each pair of disjoint closed sets *A*and *B* of *X* there exist pre-open(resp. semi-open,semi-pre-open) sets U& V for which $A \subseteq$Uand $B\subseteq$V such that U ∩ V = ϕ.

(b) A space (X,T) is said to be almost p-normal(resp. almost s-normal, almost β-normal) if for each closed set *A* and each regular closed set B such that U ∩ V = ϕ,there exist disjoint pre-open(resp. semi-open, semi-pre open) sets U&V such that $A \subseteq$Uand $B\subseteq$V .

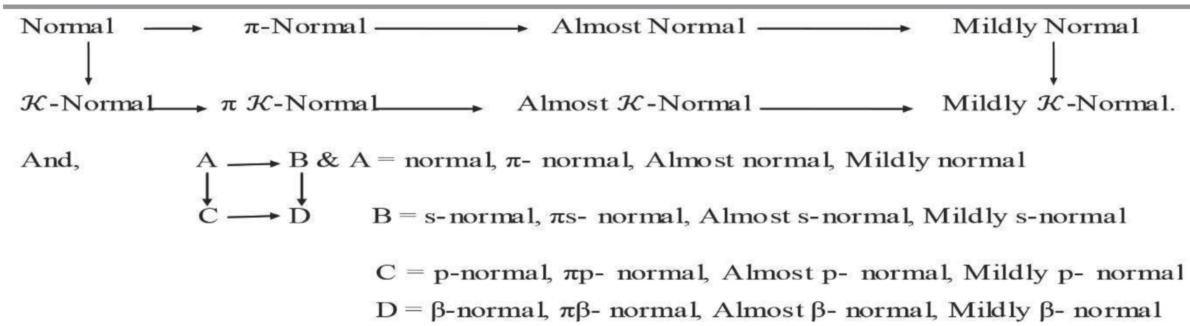
(c) A space (X,T) is said to be mildly p-normal (resp. mildly s-normal,mildly β-normal) if for each pair of disjoint regular closed sets *A* and *B* of *X* there exist pre-open (resp. semi-open, semi-pre open)sets U&V in the manner $A \subseteq$Uand $B\subseteq$V such that U ∩ V = ϕ.

(d) A space (X,T) is said to be π p-normal (resp. π s-normal, πβ-normal) if for each pair of disjoint closed sets *A* and *B* on of which is π-closed ,there exist disjoint pre-open (resp. semi-open, semi-pre open)sets U&V in the manner $A \subseteq$Uand $B\subseteq$V such that U ∩ V = ϕ.

Πβ-Normal Space: This section begins with the definition of Πβ-normality being motivated by the concept of π-normality.

Definition (2.1): A space (X,T) is said to be π p-normal (resp. π s-normal, πβ-normal) if for each pair of disjoint closed sets *A* and *B* on of which is π-closed ,there exist disjoint β-sets U&V such that $A \subseteq$Uand $B\subseteq$V .

The following is the implications diagram connecting the sorts of normal spaces indicated in definitions (1.3)&(1.4) &(2.1):



None of the above implications is reversible.

Example (2.2):

1. If X = {a, b, c, d} and T = {ϕ, {c}, {d}, {b, c}, {c, d},{a, c, d},{b, c, d}, X}. Then the space (X, T) is β-normal but not p-normal.

2. If X = {a, b, c, d, e} and _ = {ϕ_, {c, d}, {a, c, d}, {b, c, d}, {c, d, e}, {a, b, c, d}, {a, c, d, e}, {b, c, d, e}, X}.
Then the space (X, _T) is β-normal but not s-normal.

3. If X = {a,b,c,d} and T = {ϕ, {b,d},{a,b,d},{b,c,d},X}, Now,(X,T) is p-normal. But not normal.

4. If X = {a,b,c} ,T = {ϕ,X,{a,b},{a,c}}, then (X,T) is πβ-normal But (X,T) is not β-normal.

The following lemmas are enunciated as they are essential parts for the counterexamples about the other implications:

Lemma (2.3): If D be a dense subset of a space(X,T), then D is β-open.

Lemma (2.4): If D be a dense set & A is a closed set in a space (X, T), then D∪A is β-open set.

Lemma (2.5): If D be a dense set & A is a closed set in a space (X, T), then D\A is a β-open set.

Lemma (2.6): For any two disjoint closed sets A & B in a space(X, T), the sets U = (D∩ A^c)∪B & V = (D∩ B^c)∪A are β-open sets where D is a dense set in X.

Theorem (2.7): If D & E are disjoint dense subsets in a space (X,T), then (X,T) is β-normal and so πβ-normal.

Example (2.8)

(i) The co-finite topology on the set R of real numbers is a πβ-normal space but not normal.

(ii) If R stand for the set of real numbers & $T_{\sqrt{2}}$ = { A : A ⊆ R and A = ϕ or $\sqrt{2}$ ∈A }, then (R, $T_{\sqrt{2}}$) is the particular point topological space which is πβ-normal space but not β-normal.

Characterization of πβ-normality: Some characterizations of πβ-normality can be projected through the following theorem.

Theorem (2.9): For a space (X,T) the following are equivalent:

a) (X ,T) is πβ-normal space.

b) If U is an open set U and V is π-open set whose union is X, there exist β- closed sets A and B such that A ⊆U, B⊆ V & A∪ B = X.

c) For every closed set A and every π-open set B such that A ⊆B, there exists a β-open set V such that A⊆ V⊆ β-cl(V) ⊆ B.

Proof. The proof emerges in a natural way as motivated by [9].

Topological property: In order to highlight the topological property of πβ-normality, the following theorem stands as the help.

Theorem (2.10): If f: (X,T)→ (Y,σ) is an open & continuous function, then the image of a β-open set is β-open.

Theorem (2.11): πβ-normality is a topological property.

Proof: In order to show that πβ-normality is a topological property, one has to prove that the homeomorphic image of a πβ-normal space is a πβ-normal space.

Let f: (X,T)→ (Y,σ) be a one-one onto, an open & continuous function from a πβ- normal space (X,T) to another space (Y,σ). We need to show that f(X) = Y is also a πβ- normal space. Let A & B be a pair of disjoint closed sets in (Y,σ) such that A is π-closed. Obviously, the continuing of f provides that $f^{-1}(A)$ is π-closed & $f^{-1}(B)$ is closed in X such that $f^{-1}(A)$ ∩ $f^{-1}(B)$ = ϕ.

Now , the πβ – normality of (X,T), there exist β-open sets U &V of X in the manner that $f^{-1}(A)$ ⊆ U , $f^{-1}(A)$ ⊆ V and U ∩ V = ϕ.

Since, f is an open, continuous one –to one function hence, A ⊆ f(U) , B ⊆ f(V) and f(U) ∩ f(V)= ϕ. Using the theorem (2.10), we observe that f(U) & f(V) are β –open sets as U & V are β-open sets and f is an open, continuous function.

Thus, for a pair of disjoint closed sets A & B of (Y,σ) where A is π-closed, there exist disjoint β-open sets f(U) & f(V) in (Y,σ) such that A ⊆ f(U) , B ⊆ f(V) . This provides that (Y,σ) is a πβ-normal space.

Hereditary property: The following lemmas are useful and necessary for the analysis of the hereditary property of a πβ-normal space.

Lemma (2.12): If M be a closed domain (i.e. regular closed) subspace of a space X and A is β-closed in X, then A∩ M is a β-closed set in M.

Lemma (2.13): If (M, T_M) is a closed domain subspace of a space (X,T) , then A∩ M is a β-open set in (M, T_M) whenever A is a β-open set in (X,T).

Theorem (2.14) :πβ –Normality is a hereditary property with respect to closed domain subspaces.

Proof: the proof becomes so natural when we use the above lemmas (2.12)&(2.13).

Corollary (2.15): Since, every closed and open (clopen) set in a space is a regular closed set i.e.a closed domain, hence, every clopen subspace of a πβ-normal space is a πβ-normal space.

Conclusion: πβ-normality, being a weaker version of β-normality, has been introduced. It has been shown that πβ-normality is a topological property as well as hereditary property with regard to closed domain spaces. Characterization as well as preservation theorem for πβ-normality has been established. Some counter examples and the criteria for the space to bear πβ-normality in terms of disjoint dense subset have been derived.

Surely the literature content for the πβ-normality is a motivation to analyse $\pi\gamma$ -normality with fundamental properties which creates the future scope of the study.

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