CHAINED PROPERTIES OF TERNARY SEMI GROUPS

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Abstract: In this paper the terms, Chained ternary semi groups is introduced. It is proved that A ternary semi group T is semi primary if and only if prime ideals of T form a chain under set inclusion. It is also proved that . in a duo semi primary ternary semi group T, the globally idempotent principal ideals form a chain under set inclusion and the principal ideals of T form a chian iff ideals in T form a chain. It is also proved that , If T is a commutative regular ternary semi group with identity 1. Then principal ideals of T form a chain if and only if idempotents of T form a chain under natural ordering. It is also proved that, If T is a commutative semi simple ternary semi group with identity 1, then 1. Every ideal in T is a prime ideal, 2. T is a primary ternary semi group, 3. T is a left primary ternary semi group, 4. T is a lateral primary ternary semi group, 5. T is a right primary ternary semi group, 6. T is a semi primary ternary semi group, 7. Prime ideals of T form a chain, 8. Ideals of T form a chain, 9. Principal ideals of T form a chain, 10. Idempotents of T forms a chain under natural ordering are equivalent. It is also proved that, If T is a ternary semi group, then 1. Every ideal in T is a prime ideal., 2. T is a semi simple and ideals of T form a chain., 3. T is a semi simple and prime ideals of T form a chain are equivalent and also prove that If T is a commutative semi simple ternary semi group. Then 1. Every ideal in T is a prime ideal., 2. T is a regular primary ternary semi group., 3. T is a regular semi primary ternary semi group. 4. T is a regular ternary ternary semi group and idempotents of T form a chain under natural ordering are equivalent.

Keywords: About four key words or phrases in alphabetical order, separated by commas.

1. Introduction: The algebraic theory of semi groups was widely studied by CLIFFORD [2], [3], PETRICH [4] and LJAPIN [5]. The ideal theory in general semi groups was developed by ANJANEYULU [1]. In this paper we introduce the notions of chained ternary semi groups. We obtained some results on Chained properties of ternary semi groups.

2. Preliminaries:

Definition 2.1: A system S = (S, .), where S is a nonempty set and **.** is an associative binary operation on S, is called a ***semi group.***

Definition2.2: A semi group S is said to be ***commutative*** provided *ab* = *ba* forall *a, b*∈ S.

Definition 2.3: A semi group S is said to be ***normal*** provided *aS* = *Sa* for all *a* ∈ S.

Definition 2.4: An element *a* of a semi group S is said to be a ***two sided identity*** or an***identity***provided *as* =*sa*= *s* for all *s*∈ S.

Definition 2.5: An element *a* of a semi group S is said to be **a *two sided zero***or ***zero*** of S provided provided *sa* =*as* = *a* for all *s*∈ S.

Definition 2.6 : A nonempty subset A of a semi group S is said to be a***left ideal*** ***(rightideal)*** of S provided SA⊆A (AS⊆A).

Definition 2.7: A nonempty subset A of a semi group S is said to be a ***two sided ideal*** or ***ideal*** of S provided it is both a left ideal and a right ideal of S.

Definition2.8: An ideal A of a semi group S is said to be a ***proper ideal*** of S provided A ≠ S.

Definition 2.9: An ideal A of a semi group S is said to be a ***trivial ideal*** of S provided S\A is singleton.

Theorem 2.10 : The nonempty intersection of any family of ideals of a semi group S is an ideal of S.

Theorem 2.11 : The union of any family of ideals of a semi group S is an ideal of S.

Definition 2.12 : Let S be a semi group. The intersection of all ideals of S containing a nonempty set A is called the ***ideal genetated by A.***It is denoted by < A >.

Definition 2.13 : An ideal A of a semi group S is said to be a ***principal ideal*** provided A is an ideal generated by single element set. If an ideal A is generated by *a,* then A is denoted as <*a*> or J[*a*].

Definition 2.14 : An ideal A of a semi group S is said to be a ***maximal ideal*** provided A is a proper ideal of S and is not properly contained in any proper ideal of S.

Definition 2.15 : An ideal A of a semi group S is said to be a ***minimal ideal*** provided A does not contain any ideal of S properly.

Definition 2.16: An ideal A of a semi group S is said to be a ***completely prime ideal*** provided *x, y*∈ S, *xy* ∈ A, implies either *x* ∈ A or *y* ∈ A.

Definition 2.17 : An ideal A of a semi group S is said to be a ***prime ideal*** provided X,Y are ideals of S, XY ⊆A implies either X⊆A or Y ⊆A.

Theorem 2.18 : Every completely prime ideal of a semi group is prime.

Definition 2.19 : If A is an ideal of a semi group S, then the intersection of all prime ideals containing A is called ***prime radical*** or simply ***radical*** of A and it is denoted by $\sqrt{A}$ or ***rad*** **A**.

Definition 2.20: An ideal A of a semi group S is said to be ***completely semiprime*** provided *x*∈S, x^n∈ A for some natural number *n* implies *x* ∈ A.

Theorem 2.21 : The nonempty intersection of any family of completely prime ideals of a semi group is completely semiprime.

Theorem 2.22 : If S is a globally idempotent semi group then every maximal ideal M of S is a prime ideal of S.

Definition 2.23 : An ideal A of a semi group S is said to be ***semiprime*** provided X is an ideal of S, $X^n \subseteq A$ for some natural number *n* implies X ⊆ A.

Theorem 2.24: An ideal Q of a semi group S is a semiprime ideal of S iff √Q = Q.

Theorem 2.25 : If A, B and C are any three ideals of a ternary semi group T , then

i) $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$

ii) if A ∩ B∩ C ≠ ∅ then $\sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$

iii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

Definition2.25: An ideal A of a semi group S is said to be ***pseudo symmetric*** provided $x, y \in S$, $xy \in A$ implies $xsy \in A$ for all $s \in S$.

Theorem 2.26: Let A be an ideal of a semi group S. Then A is completely prime iff A is prime and pseudo symmetric.

Definition 2.27 : A semi group S is said to be ***pseudo symmetric*** provided every ideal in S is a pseudo symmetric ideal.

Definition 2.28 : An ideal A in a semi group S is said to be ***semipseudo symmetric*** provided for any natural number *n*, $x \in S$, $x^n \in A$, $\Rightarrow \langle x^n \rangle \subseteq A$.

Theorem 2.29 : Every pseudo symmetric ideal of a semi group is a semipseudo symmetric ideal.

Theorem 2.30 : If A is an ideal of a semi group S, then the following are equivalent.

1) A is completely prime.
2) A is prime and pseudo symmetric.
3) A is prime and semipseudo symmetric.

Theorem 2.31 : Let A be a semipseudo symmetric ideal of a semi group S. Then the following are equivalent.

1) A_1=The intersection of all completely prime ideals of S containing A.
2) A_1^1 = The intersection of all minimal completely prime ideals of S containing A.
3) A_1^{11} = The minimal completely semiprime ideal of S relative to containing A.
4) $A_2=\{x\in S: x^n \in A$ for some natural numbern}
5) A_3= The intersection of all prime ideals of S containing A.
6) A_3^1= The intersection of all minimal prime ideals of S containing A.
7) A_3^{11} = The minimal semiprime ideal of S relative to containing A.
8) $A_4 = \{x\in S : \langle x\rangle^n \subseteq A$ for some natural number n}

Corollary 2.32 : An ideal Q of a semi group S is a semiprime ideal iff Q is the intersection of all prime ideals of S contains Q.

Theorem 2.33: If T is a commutative regular ternary semi group with identity *e*. Then every principal ideal of T is generated by an idempotent.

Theorem 2.33 : If A is a semiprime ideal of a ternary semi group T, then the following are equivalent.

1) A is a prime ideal
2) A is a primary ideal
3) A is a left primary ideal
4) A is a lateral primary ideal
5) A is a right primary ideal
6) A is a semi primary ideal.

Theorem 2.34: Let T be a commutative ternary semi group with identity *1*. If $a\in T$, then the following are equivalent.

1) A is regular .
2) A is left regular .
3) A is right regular.
4) A is lateral regular.
5) A is semi simple.

3. Chained properties of ternary semi group:

Definition 3.1 : A ternary semi group T is said to be a ***chained ternary semi group*** if the ideals in T are linearly ordered by set inclusion.

Theorem 3.2 : A ternary semi group T is semi primary if and only if prime ideals of T form a chain under set inclusion.

***Proof* :** Suppose that T is a semi primary ternary semi group. Let A, B be three prime ideals of T. By theorem 2.25, $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B} = A \cap B$. Therefore by theorem 2.24, $A \cap B$ is semiprime. Since T is a semi primary ternary semi group it follows that $A \cap B$ is semi primary. By theorem 2.31, $A \cap B$ is prime. Suppose if possible A⊈B and B⊈A. Then there exists $a \in A\backslash B$, $b \in B\backslash A$. Now $\langle a\rangle\langle b\rangle\langle b\rangle \subseteq A \cap B$ and $a, b \notin A \cap B$. It is a contradiction. Therefore prime ideals of T forms a chain under set inclusion. Conversely, suppose that prime ideals of T form a chain under set inclusion For every ideal A, $\sqrt{A} = \cap P_\alpha$, where intersection is over all prime ideals P_α containing A yields $\sqrt{A} = P_\alpha$ for some α. So that A is a semi primary ideal. Therefore T is a semi primary ternary semi group.

Theorem 3.3 : If T is a duo semi primary ternary semi group, then globally idempotent principal ideals form a chain under set inclusion.

***Proof* :** Suppose that $\langle x\rangle$, $\langle y\rangle$ are two globally idempotent principal ideals in T. Since T is a semi primary ternary semi group, we have $\sqrt{\langle x \rangle}$and

$\sqrt{< y >}$ are prime ideals of T. By theorem 3.2, either $\sqrt{< x >}\subseteq\sqrt{< y >}$ or $\sqrt{< y >}\subseteq \sqrt{< x >}$.

If $\sqrt{< x >}\subseteq\sqrt{< y >}$, then $x \in\sqrt{< y >}$ and hence $x^n\in <y>$.

Therefore $<x>^n\subseteq<y>$ for some odd natural number *n*.

Since <*x*> is a globally idempotent ideal, $<x>^n = <x>$ and hence <*x*>⊆<*y*>.

Similarly we can show that if $\sqrt{< y >}\subseteq\sqrt{< x >}$, then <*y*>⊆ <*x*>.

Therefore globally idempotent principal ideals forms a chain under set inclusion.

Theorem 3.4 : Let T be a ternary semi group. Then the principal ideals of T form a chian iff ideals in T form a chain .

Proof : Suppose that principal ideals of T form a chain. Let A, B be two ideals of T.

Suppose if possible A ⊈ B and B ⊈ A. Then there exists *a*∈ A\B and *b*∈ B\A. Now *a*∈ A ⇒<*a*>⊆ A and *b* ∈ B ⇒<*b*>⊆ B. Since principal ideals form a chain, either <*a*>⊆<*b*> or <*b*>⊆<*a*>. If <*a* >⊆<*b*> then *a*∈<*b*>⊆ B, which is not true. If <*b* >⊆<*a*> then *b*∈<*a*>⊆ A, which is not true. It is contradiction. Hence either A ⊆ B or B ⊆ A. Therefore ideals of T form a chain.

Conversely suppose that ideals of T form a chain. Then clearly principal ideals of T form a chain.

Theorem 3.5 : Let T be a commutative regular ternary semi group with identity 1. Then principal ideals of T form a chain if and only if idempotents of T form a chain under natural ordering.

Proof **:** Since T is a commutative regular ternary semi group with identity 1.

Suppose that principal ideals of T form a chain. Let *e* and *f* be two idempotents in S. Then <*e*> , <*f*> are globally idempotent principal ideal of T.

Since principal ideals of T form a chain, <*e*>⊆<*f*> or <*f*>⊆<*e* >.

If <*e*>⊆<*f*>, then e ∈ <*f*>. Since T is a commutative ternary semi group, $e \in<f>= f\,TT = TTf$, implies that $e = fst = pqf$ for some *s, t* ∈ S.Now $eff = pqfff = pqf = e$ and $ffe = fffst = fst = e$.

Therefore $e \le f$.Similarly if <*f*>⊆<*e* > , then we get $f \le e$.

Therefore the idempotents in T form a chain under natural ordering.

Conversely suppose that idempotents in T form a chain under natural ordering. Let <*a*> , <*b*> be two principal ideals of S. By theorem 2.33, <*a*> = <*e*> and <*b*> = <*f*> for some idempotent elements *e*, *f* in S.

Since *e* and *f* are idempotents in T and the idempotents in T form a chain under natural ordering , either $e\le f$ or $f\le e$. If $e\le f$, then $e = ffe = eff\in<f>$ and hence <*e*>⊆<*f*> i.e.,<*a*>⊆<*b*>. If $f\le e$, then $f = eef = fee \in<e>$ and hence <*f*>⊆<*e*> i.e.,<*b*>⊆<*a*>. Therefore principal ideals of T form a chain.

Theorem 3.6 : If T is a semi simple ternary semi group, then the following are equivalent.

1. Every ideal in T is a prime ideal.
2. T is a primary ternary semi group.
3. T is a left primary ternary semi group.
4. T is a lateral primary ternary semi group.
5. T is a right primary ternary semi group.
6. T is a semi primary ternary semi group.
7. Prime ideals of T form a chain

Proof **:** Suppose that T is a semi simple ternary semi group.

Therefore $<x> = <x>^3$for all *x*∈ T. Let A be any ideal of T. Suppose that *x*∈ T and $<x>^3\subseteq$A. Then $x\in <x>^3 \subseteq$ A and hence *x*∈ A . Therefore A is a semiprime ideal of T. Therefore every ideal of T is a semiprime ideal of T. By theorem 2.33, (1) to (6) are equivalent. By the theorem 3.2, (6) and (7) are equivalent. Hence (1) to (7) are equivalent.

Theorem 3.7 : If T is a commutative semi simple ternary semi group with identity 1, then the following are equivalent.

1. Every ideal in T is a prime ideal.
2. T is a primary ternary semi group.
3. T is a left primary ternary semi group.
4. T is a lateral primary ternary semi group.
5. T is a right primary ternary semi group.
6. T is a semi primary ternary semi group.
7. Prime ideals of T form a chain.
8. Ideals of T form a chain.
9. Principal ideals of T form a chain.
10. Idempotents of T forms a chain under natural ordering.

Proof **:** By theorem 3.6, (1) to (7) are equivalent.

By theorem 3.4, (8) and (9) are equivalent. By theorem 3.5, (9) and (10) are equivalent.

Clearly, (8) ⇒ (7) is true.

By theorem 3.3, (6) ⇒ (10) is true.

Therefore (1) to (10) are equivalent.

Theorem 3.8 : Every ideal of a ternary ternary semi groupT is prime if and only if T is semi simple primary ternary semi group.

Proof **:** Suppose that every ideal of T is prime. Then by theorem 3.2, every ideal of T is a primary ideal. Therfore T is a primary ternary semi group. Let *a*∈ T.

Now $<a><a><a>\subseteq<a>^3$ and since $<a>^3$ is prime, implies that $<a>\subseteq<a>^3$.

So *a* is semi simple and therefore T is a semi simple primary ternary semi group.Conversely suppose that

T is a semi simple primary ternary semi group. By thorem 3.2, every ideal in T is prime.

Theorem 3.9 : If T is a ternary semi group, then the following are equivalent.

1. Every ideal in T is a prime ideal.
2. T is a semi simple and ideals of T form a chain.
3. T is a semi simple and prime ideals of T form a chain.

Proof **:** (1) ⇒ (2) : Suppose that every ideal of a ternary semi group T is prime. By theorem 3.3, T is a semi simple primary semi group. By theorem 3.2, ideals in T form a chain . Hence T is a semi simple ternary semi group and ideals in T form a chain.

(2) ⇒ (3) is clear.

(3) ⇒ (1) : Suppose that T is a semi simple and prime ideals of T form a chain. Then By theorem 3.2, every ideal of T is prime.

Theorem 3. 10 : If T is a commutative semi simple ternary semi group. Then the following are equivalent.

1. Every ideal in T is a prime ideal.
2. T is a regular primary ternary semi group.
3. T is a regular semi primary ternary semi group.
4. T is a regular ternary ternary semi group and idempotents of T form a chain under natural ordering.

Proof **:**(1) ⇒ (3) : Suppose that every ideal of T is prime. Let A be an ideal of T. Then A is prime. By theorem 2.33, T is primary ternary semi group.

Let $a\in$ T. Now $<a><a><a>\subseteq<a>^3$ and $<a>^3$ is a prime ideal of T, we have $<a>\subseteq<a>^3$ and hence $a\in<a>^3$. So a is semi simple for all $a\in$ T. Since T is commutative semi simple, by theorem 2.34, T is regular. Hence T is a regular primary ternary semi group.

(2)⇒(3) : Suppose that T is a regular primary ternary semi group. By theorem 3.2, T is semi primary ternary semi group. Therefore T is a regular semi primary ternary semi group.

(3)⇒(4) : Suppose that T is regular semi primary ternary semi group. Since T is semi primary, by theorem 3.7, the idempotents of T forms a chain under natural ordering. Hence T is regular and idempotents of T forms a chain under natural ordering.

(4)⇒(1) : Suppose that T is a regular terinary ternary semi group and idempotents of T form a chain under natural ordering. By theorem 3.7, every ideal of T is prime.

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