

**SOME FIXED AND COMMON FIXED POINT THEOREM IN 2-BANACH SPACE**

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*Abstract:* In this paper we prove fixed point and common fixed point Theorem in non-expansive and existence mappings using Banach space for new rational expression. Which generalize the well known result.

*Keywords:* Banach space, Common fixed point, fixed point, rational expression.

**I. Introduction:** Fixed point has drawn the attentions of the authors working in non-linear analysis, the study of non-expansive mapping and the existence of fixed point. The non-expansive mappings include contraction as well as contractive mappings. Browder [1] was the first mathematician to study non-expansive mappings; he applied these results for proving the existence of solutions of certain integral equations. Browder [1], Gohde [3] and Kirk [11] have independently proved a fixed point theorem for non-expansive mappings defined on a closed bounded and convex subset of a uniformly convex Banach space and in the spaces with richer generalizations of non-expansive mappings, prominent being Datson [12], Emmanuele [2], Goebel [4], Goebel and Zlotkiewicz [5], Iseki [6], Sharma & Rajput [10], Singh and Chatterjee [8]. They have derived valuable results with non-contraction mapping in Banach space. It is well known that a Banach space is a linear space which is also in a special way a complete metric space. The combination of algebraic and metric structures opens up the possibility of studying linear transformation of one Banach space into another which has the additional property of being continuous. A normed linear space is a linear space N in which to each vector z, there corresponds a real number denoted by ||x|| and called the norm of x in such a manner that

- (i) ||x|| ≥ 0 and ||x|| = 0 ⇔ x = 0
- (ii) ||x + y|| ≤ ||x|| + ||y||
- (iii) ||αx|| = |α| ||x|| β is real

The non negative real number ||x|| is to be thought of as the length of vector x. If we regard ||x|| as a real function defined on N. It is easy to verify that the normed function is called norm on N. It is easy to verify that the normed linear space N is a metric space w.r.to the metric d defined by d(x, y) = ||x - y||. A Banach space is a complete normed linear space. Let X be a Banach space on the complex field numbers E, and E be a non empty convex subset of X. A map T : C → C is said to be an mean nonexpansive mapping. A mapping T on a subset E of a Banach space X is called a nonexpansive mapping if ||Tx - Ty|| ≤ ||x - y|| for all x, y ∈ E.

Our object in this chapter is to prove some fixed and common fixed point theorems in non-expansive and existence mappings using Banach space.

**Definition:** Let K be a nonempty subset of a real normed space E. Let T be a self-mapping of K. Then T is said to be non expansive if ||Tx - Ty|| ≤ ||x - y|| for all x, y ∈ K

**II. MAIN RESULT:**

**Theorem 2.1:** Let T be a mapping of a Banach spaces into itself. If T satisfies the following conditions:

- 1. T<sup>2</sup> = I, where I is identity mapping.
- 2.

$$\|Tx - Ty\| \geq \alpha \left[ \frac{\|x - Tx\|^2 \|y - Ty\| + \|x - Tx\| \|y - x\|^2}{2 \|x - Tx\| \|y - Ty\|} \right] + \beta \left[ \frac{\|x - Tx\|^2 \|x - Ty\| + \|y - x\|^2 \|x - Tx\|}{\|x - Tx\| \|y - Ty\| + \|x - Tx\| \|y - x\|} \right] + \gamma \left[ \frac{\|x - Tx\| + \|y - Ty\|}{2} \right] + \delta \left[ \frac{\|x - Ty\| + \|y - Tx\|}{2} \right] + \eta \|x - y\| + \xi \left[ \frac{\|x - Tx\| \|y - Ty\|}{\|x - y\|} \right]$$

For every x ≠ y and 20α + 4β + 8γ + 8δ + 4η + 16ξ > 8. Then T has a unique fixed point in X.

**Proof:** Suppose x is any point in Banach space. Taking y = 1/2 (T + I)x, z = T(y), v = 2y - z, we have

$$\|z - x\| = \|Ty - T^2x\| = \|Ty - T(Tx)\|$$

$$\|z - x\| \geq \alpha \left[ \frac{\|y - Ty\|^2 \|Tx - T(Tx)\| + \|y - Ty\| \|Tx - y\|^2}{2 \|y - Ty\| \|Tx - T(Tx)\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \|y - T(Tx)\| + \|Tx - y\|^2 \|y - Ty\|}{\|y - Ty\| \|Tx - T(Tx)\| + \|y - Ty\| \|Tx - y\|} \right] + \gamma \left[ \frac{\|y - Ty\| + \|Tx - T(Tx)\|}{2} \right] + \delta \left[ \frac{\|y - T(Tx)\| + \|Tx - Ty\|}{2} \right] + \eta \|y - Tx\| + \xi \left[ \frac{\|y - Ty\| \|Tx - T(Tx)\|}{\|y - Tx\|} \right]$$

$$\|z - x\| \geq \alpha \left[ \frac{\|y - Ty\|^2 \|Tx - x\| + \|y - Ty\| \|Tx - y\|^2}{2 \|y - Ty\| \|Tx - x\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \|y - x\| + \|Tx - y\|^2 \|y - Ty\|}{\|y - Ty\| \|Tx - x\| + \|y - Ty\| \|Tx - y\|} \right]$$

$$\begin{aligned}
 & +\gamma \left[ \frac{\|y - Ty\| + \|Tx - x\|}{2} \right] + \delta \left[ \frac{\|y - x\| + \|Tx - y\| + \|y - Ty\|}{2} \right] + \eta \|y - Tx\| + \xi \left[ \frac{\|y - Ty\| \|Tx - x\|}{\|y - Tx\|} \right] \\
 \|z - x\| \geq & \alpha \left[ \frac{\|y - Ty\|^2 \|Tx - x\| + \|y - Ty\| \left\| Tx - \frac{1}{2}(T + I)x \right\|^2}{2\|y - Ty\| \|Tx - x\|} \right] \\
 & + \beta \left[ \frac{\|y - Ty\|^2 \left\| \frac{1}{2}(T + I)x - x \right\| + \left\| Tx - \frac{1}{2}(T + I)x \right\|^2 \|y - Ty\|}{\|y - Ty\| \|Tx - x\| + \|y - Ty\| \left\| Tx - \frac{1}{2}(T + I)x \right\|} \right] + \gamma \left[ \frac{\|y - Ty\| + \|Tx - x\|}{2} \right] \\
 & + \delta \left[ \frac{\left\| \frac{1}{2}(T + I)x - x \right\| + \left\| Tx - \frac{1}{2}(T + I)x \right\| + \|y - Ty\|}{2} \right] + \eta \left\| \frac{1}{2}(T + I)x - Tx \right\| + \xi \left[ \frac{\|y - Ty\| \|Tx - x\|}{\left\| \frac{1}{2}(T + I)x - Tx \right\|} \right] \\
 \|z - x\| \geq & \alpha \left[ \frac{\|y - Ty\|^2 \|Tx - x\| + \|y - Ty\| \frac{1}{4} \|Tx - x\|^2}{2\|y - Ty\| \|Tx - x\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \frac{1}{2} \|Tx - x\| + \frac{1}{4} \|Tx - x\|^2 \|y - Ty\|}{\|y - Ty\| \|Tx - x\| + \|y - Ty\| \frac{1}{2} \|Tx - x\|} \right] \\
 & + \gamma \left[ \frac{\|y - Ty\| + \|Tx - x\|}{2} \right] + \delta \left[ \frac{\frac{1}{2} \|Tx - x\| + \frac{1}{2} \|Tx - x\| + \|y - Ty\|}{2} \right] + \eta \frac{1}{2} \|x - Tx\| + \xi \left[ \frac{\|y - Ty\| \|Tx - x\|}{\frac{1}{2} \|x - Tx\|} \right] \\
 \|z - x\| \geq & \alpha \left[ \frac{\|y - Ty\| \|Tx - x\| \left( \|y - Ty\| + \frac{1}{4} \|Tx - x\| \right)}{2\|y - Ty\| \|Tx - x\|} \right] + \beta \left[ \frac{\|y - Ty\| \|Tx - x\| \left( \frac{1}{2} \|y - Ty\| + \frac{1}{4} \|x - Tx\| \right)}{\frac{3}{2} \|y - Ty\| \|Tx - x\|} \right] \\
 & + \gamma \left[ \frac{\|y - Ty\| + \|Tx - x\|}{2} \right] + \delta \left[ \frac{\|Tx - x\| + \|y - Ty\|}{2} \right] + \eta \|x - Tx\| + 2\xi \|y - Ty\| \\
 \|z - x\| \geq & \frac{\alpha}{2} \|y - Ty\| + \frac{\alpha}{8} \|Tx - x\| + \frac{\beta}{3} \|y - Ty\| + \frac{\beta}{6} \|Tx - x\| + \frac{\gamma}{2} \|y - Ty\| + \frac{\gamma}{2} \|Tx - x\| + \frac{\delta}{2} \|Tx - x\| + \\
 & \frac{\delta}{2} \|Ty - y\| + \frac{\eta}{2} \|Tx - x\| + 2\xi \|y - Ty\| \\
 \|z - x\| \geq & \left( \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi \right) \|y - Ty\| + \left( \frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right) \|Tx - x\| \tag{2.1} \\
 \|v - x\| = & \|2y - z - x\| = \left\| 2 \frac{1}{2}(T + I)x - Ty - x \right\| = \|Tx - Ty\| \\
 \|v - x\| \geq & \alpha \left[ \frac{\|y - Ty\|^2 \|x - Tx\| + \|y - Ty\| \|x - y\|^2}{2\|y - Ty\| \|x - Tx\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \|y - Tx\| + \|x - y\|^2 \|y - Ty\|}{\|y - Ty\| \|x - Tx\| + \|y - Ty\| \|x - y\|} \right] + \\
 & \gamma \left[ \frac{\|y - Ty\| + \|x - Tx\|}{2} \right] + \delta \left[ \frac{\|y - Tx\| + \|x - Ty\|}{2} \right] + \eta \|y - x\| + \xi \left[ \frac{\|y - Ty\| \|x - Tx\|}{\|y - x\|} \right] \\
 \|v - x\| \geq & \alpha \left[ \frac{\|y - Ty\|^2 \|x - Tx\| + \|y - Ty\| \left\| x - \frac{1}{2}(T + I)x \right\|^2}{2\|y - Ty\| \|x - Tx\|} \right] + \\
 & \beta \left[ \frac{\|y - Ty\|^2 \left\| \frac{1}{2}(T + I)x - Tx \right\| + \left\| x - \frac{1}{2}(T + I)x \right\|^2 \|y - Ty\|}{\|y - Ty\| \|x - Tx\| + \|y - Ty\| \left\| x - \frac{1}{2}(T + I)x \right\|} \right] + \gamma \left[ \frac{\|y - Ty\| + \|x - Tx\|}{2} \right] + \\
 & \delta \left[ \frac{\left\| \frac{1}{2}(T + I)x - Tx \right\| + \left\| x - \frac{1}{2}(T + I)x \right\| + \|y - Ty\|}{2} \right] + \eta \left\| \frac{1}{2}(T + I)x - x \right\| + \xi \left[ \frac{\|y - Ty\| \|x - Tx\|}{\left\| \frac{1}{2}(T + I)x - x \right\|} \right]
 \end{aligned}$$

$$\|v - x\| \geq \alpha \left[ \frac{\|y - Ty\|^2 \|x - Tx\| + \|y - Ty\| \frac{1}{4} \|x - Tx\|^2}{2 \|y - Ty\| \|x - Tx\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \frac{1}{2} \|x - Tx\| + \frac{1}{4} \|x - Tx\|^2 \|y - Ty\|}{\|y - Ty\| \|x - Tx\| + \|y - Ty\| \frac{1}{2} \|x - Tx\|} \right]$$

$$+ \gamma \left[ \frac{\|y - Ty\| + \|x - Tx\|}{2} \right] + \delta \left[ \frac{\frac{1}{2} \|x - Tx\| + \frac{1}{2} \|x - Tx\| + \|y - Ty\|}{2} \right] + \eta \frac{1}{2} \|Tx - x\| + \xi \left[ \frac{\|y - Ty\| \|x - Tx\|}{\frac{1}{2} \|Tx - x\|} \right]$$

$$\|v - x\| \geq \alpha \left[ \frac{\|y - Ty\| \|x - Tx\| (\|y - Ty\| + \frac{1}{4} \|x - Tx\|)}{2 \|y - Ty\| \|x - Tx\|} \right] + \beta \left[ \frac{\|y - Ty\| \|x - Tx\| (\frac{1}{2} \|y - Ty\| + \frac{1}{4} \|x - Tx\|)}{\frac{3}{2} \|y - Ty\| \|x - Tx\|} \right] +$$

$$\gamma \left[ \frac{\|y - Ty\| + \|x - Tx\|}{2} \right] + \delta \left[ \frac{\|x - Tx\| + \|y - Ty\|}{2} \right] + \frac{\eta}{2} \|Tx - x\| + 2\xi \|y - Ty\|$$

$$\|v - x\| \geq \frac{\alpha}{2} \|y - Ty\| + \frac{\alpha}{8} \|x - Tx\| + \frac{\beta}{3} \|y - Ty\| + \frac{\beta}{6} \|x - Tx\| + \frac{\gamma}{2} \|y - Ty\| + \frac{\gamma}{2} \|x - Tx\| + \frac{\delta}{2} \|x - Tx\| +$$

$$\frac{\delta}{2} \|Ty - y\| + \frac{\eta}{2} \|x - Tx\| + 2\xi \|y - Ty\|$$

$$\|v - x\| \geq \left( \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi \right) \|y - Ty\| + \left( \frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right) \|x - Tx\| \tag{2.2}$$

Now, by (2.1) and (2.2), we get

$$\|z - v\| \geq \|z - x\| + \|x - v\|$$

$$\|z - v\| \geq 2 \left( \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi \right) \|y - Ty\| + 2 \left( \frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right) \|x - Tx\| \tag{2.3}$$

Also,

$$\|z - v\| = \|Ty - (2y - z)\| = \|Ty - 2y + Ty\| = 2 \|Ty - y, u\| \tag{2.4}$$

Therefore by (2.3) and (2.4), we have

$$2 \|Ty - y, u\| \geq 2 \left( \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi \right) \|y - Ty\| + 2 \left( \frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right) \|x - Tx\|$$

$$2 \|Ty - y, u\| - 2 \left( \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi \right) \|y - Ty\| \geq 2 \left( \frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right) \|x - Tx\|$$

$$\|x - Tx\| \leq \frac{1 - \left( \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi \right)}{\left( \frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right)} \|y - Ty\|$$

$$\|x - Tx\| \leq m \|y - Ty\| \text{ As } 20\alpha + 4\beta + 8\gamma + 8\delta + 4\eta + 16\xi > 8.$$

$$\text{where } m = \frac{1 - \left( \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi \right)}{\left( \frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right)} < 1$$

Let  $R = \frac{1}{2} (T + I)$  then for every  $x \in X$ ,

$$\|R^2x - Rx\| = \|R(Rx) - Rx\| = \|Ry - y\| = \left\| \frac{1}{2} (T + I)y - y \right\| = \frac{1}{2} \|y - Ty\| < \frac{m}{2} \|x - Tx\|$$

By the definition of R we claim that  $\{R^n(x)\}$  is a Cauchy sequence in X,  $\{R^n(x)\}$  is converges to so element  $x_0$  in X. So  $\lim_{n \rightarrow \infty} \{R^n(x)\} = x_0$  So  $\{R(x_0)\} = x_0$ . Hence  $T(x_0) = x_0$ . So  $x_0$  is a fixed point of T.

**Uniqueness:** If possible let  $y_0 \neq x_0$  is another fixed point of T. Then

$$\|x_0 - y_0, u\| = \|Tx_0 - Ty_0, u\| \geq \alpha \left[ \frac{\|x_0 - Tx_0\|^2 \|y_0 - Ty_0\| + \|x_0 - Tx_0\| \|y_0 - Ty_0\|^2}{2 \|x_0 - Tx_0\| \|y_0 - Ty_0\|} \right]$$

$$+ \beta \left[ \frac{\|x_0 - Tx_0\|^2 \|x_0 - Ty_0\| + \|y_0 - Ty_0\|^2 \|x_0 - Tx_0\|}{\|x_0 - Tx_0\| \|y_0 - Ty_0\| + \|x_0 - Tx_0\| \|y_0 - Ty_0\|} \right] + \gamma \left[ \frac{\|x_0 - Tx_0\| + \|y_0 - Ty_0\|}{2} \right]$$

$$+ \delta \left[ \frac{\|x_0 - Ty_0\| + \|y_0 - Tx_0\|}{2} \right] + \eta \|x_0 - y_0\| + \xi \left[ \frac{\|x_0 - Tx_0\| \|y_0 - Ty_0\|}{\|x_0 - y_0\|} \right]$$

Putting  $T(x_0) = x_0$  and  $T(y_0) = y_0$ .

$$\|x_0 - y_0, u\| \geq \alpha \left[ \frac{\|x_0 - x_0\|^2 \|y_0 - y_0\| + \|x_0 - x_0\| \|y_0 - x_0\|^2}{2 \|x_0 - x_0\| \|y_0 - y_0\|} \right] + \beta \left[ \frac{\|x_0 - x_0\|^2 \|x_0 - y_0\| + \|y_0 - x_0\|^2 \|x_0 - x_0\|}{\|x_0 - x_0\| \|y_0 - y_0\| + \|x_0 - x_0\| \|y_0 - x_0\|} \right]$$

$$+ \gamma \left[ \frac{\|x_0 - x_0\| + \|y_0 - y_0\|}{2} \right] + \delta \left[ \frac{\|x_0 - y_0\| + \|y_0 - x_0\|}{2} \right] + \eta \|x_0 - y_0\| + \xi \left[ \frac{\|x_0 - x_0\| \|y_0 - y_0\|}{\|x_0 - y_0\|} \right]$$

$\|x_0 - y_0\| \geq (\delta + \eta) \|x_0 - y_0\|$  Therefore,  $\|x_0 - y_0\| = 0$ . Which is contradiction, so  $x_0 = y_0$ . Hence  $x_0$  is a unique fixed point in T.

**Theorem 2.2:** Let T and G be two expansion mapping of a Banach spaces X into itself. If T and G satisfies the following conditions:

1.  $T^2 = I$  and  $G^2 = I$ , where I is identity mapping.
2. T ad G commute.
- 3.

$$\|Tx - Ty\| \geq \alpha \left[ \frac{\|Gx - Tx\|^2 \|Gy - Ty\| + \|Gx - Tx\| \|Gy - Gx\|^2}{2 \|Gx - Tx\| \|Gy - Ty\|} \right]$$

$$+ \beta \left[ \frac{\|Gx - Tx\|^2 \|Gx - Ty\| + \|Gy - Gx\|^2 \|Gx - Tx\|}{\|Gx - Tx\| \|Gy - Ty\| + \|Gx - Tx\| \|Gy - Gx\|} \right] + \gamma \left[ \frac{\|Gx - Tx\| + \|Gy - Ty\|}{2} \right]$$

$$+ \delta \left[ \frac{\|Gx - Ty\| + \|Gy - Tx\|}{2} \right] + \eta \|Gx - Gy\| + \xi \left[ \frac{\|Gx - Tx\| \|Gy - Ty\|}{\|Gx - Gy\|} \right]$$

For every  $x, y \in X$  and  $\alpha, \beta, \gamma, \delta, \eta, \xi \in [0,1]$  with  $x \neq y$  and  $\|Gx - Gy\| \neq 0$  and  $\delta + \eta > 1$ . Then there exists a unique commo fixed point of T and G such that  $Tx_0 = x_0$  and  $Gx_0 = x_0$ .

**Proof :** Suppose that x is any point in Banach space X it is clear that  $(TG)^2 = I$ .

$$\|T(G^2x) - T(G^2y)\| \geq \alpha \left[ \frac{\|G(G^2x) - T(G^2x)\|^2 \|G(G^2y) - T(G^2y)\| + \|G(G^2x) - T(G^2x)\| \|G(G^2y) - G(G^2x)\|^2}{2 \|G(G^2x) - T(G^2x)\| \|G(G^2y) - T(G^2y)\|} \right]$$

$$+ \beta \left[ \frac{\|G(G^2x) - T(G^2x)\|^2 \|G(G^2x) - T(G^2y)\| + \|G(G^2y) - G(G^2x)\|^2 \|G(G^2x) - T(G^2x)\|}{\|G(G^2x) - Tx\| \|G(G^2y) - T(G^2y)\| + \|G(G^2x) - T(G^2x)\| \|G(G^2y) - G(G^2x)\|} \right]$$

$$+ \gamma \left[ \frac{\|G(G^2x) - T(G^2x)\| + \|G(G^2y) - T(G^2y)\|}{2} \right] + \delta \left[ \frac{\|G(G^2x) - T(G^2y)\| + \|G(G^2y) - T(G^2x)\|}{2} \right]$$

$$+ \eta \|G(G^2x) - G(G^2y)\| + \xi \left[ \frac{\|G(G^2x) - T(G^2x)\| \|G(G^2y) - T(G^2y)\|}{\|G(G^2x) - G(G^2y)\|} \right]$$

$$\|TG(Gx) - TG(Gy)\| \geq \alpha \left[ \frac{\|Gx - TG(Gx)\|^2 \|Gy - TG(Gy)\| + \|Gx - TG(Gx)\| \|Gy - Gx\|^2}{2 \|Gx - TG(Gx)\| \|Gy - TG(Gy)\|} \right]$$

$$+ \beta \left[ \frac{\|Gx - TG(Gx)\|^2 \|Gx - TG(Gy)\| + \|Gy - Gx\|^2 \|Gx - TG(Gx)\|}{\|Gx - TG(Gx)\| \|Gy - TG(Gy)\| + \|Gx - TG(Gx)\| \|Gy - Gx\|} \right] + \gamma \left[ \frac{\|Gx - TG(Gx)\| + \|Gy - TG(Gy)\|}{2} \right]$$

$$+ \delta \left[ \frac{\|Gx - TG(Gy)\| + \|Gy - TG(Gx)\|}{2} \right] + \eta \|Gx - Gy\| + \xi \left[ \frac{\|Gx - TG(Gx)\| \|Gy - TG(Gy)\|}{\|Gx - Gy\|} \right]$$

Taking  $Gx = p$ ,  $Gy = q$  and  $TG = R$ , whrer  $p \neq q$ , we get

$$\|Rp - Rq\| \geq \alpha \left[ \frac{\|p - Rp\|^2 \|q - Rq\| + \|p - Rp\| \|q - p\|^2}{2 \|p - Rp\| \|q - Rp\|} \right] + \beta \left[ \frac{\|p - Rp\|^2 \|p - Rq\| + \|q - p\|^2 \|p - Rp\|}{\|p - Rp\| \|q - Rq\| + \|p - Rp\| \|q - p\|} \right]$$

$$+ \gamma \left[ \frac{\|p - Rp\| + \|q - Rq\|}{2} \right] + \delta \left[ \frac{\|p - Rq\| + \|q - Rp\|}{2} \right] + \eta \|p - q\| + \xi \left[ \frac{\|p - Rp\| \|q - Rq\|}{\|p - q\|} \right]$$

It is clear by theorem (3.1) that  $R = TG$  has at least one fixed point say  $x_0$  in K that is  $R(x_0) = TG(x_0) = x_0$  and so  $T(TG)(x_0) = T(x_0)$  or  $T^2(Gx_0) = T(x_0)$  Implies that  $G(x_0) = T(x_0)$ .

Now  $\|Tx_0 - x_0\| = \|Tx_0 - T^2x_0\| = \|Tx_0 - T(Tx_0)\|$

$$\geq \alpha \left[ \frac{\|G(x_0) - T(x_0)\|^2 \|G(Tx_0) - T(Tx_0)\| + \|G(x_0) - T(x_0)\| \|G(Tx_0) - G(x_0)\|^2}{2 \|G(x_0) - T(x_0)\| \|G(Tx_0) - T(Tx_0)\|} \right]$$

$$+ \beta \left[ \frac{\|G(x_0) - T(x_0)\|^2 \|G(x_0) - T(Tx_0)\| + \|G(Tx_0) - G(x_0)\|^2 \|G(x_0) - T(x_0)\|}{\|G(x_0) - T(x_0)\| \|G(Tx_0) - T(Tx_0)\| + \|G(x_0) - T(x_0)\| \|G(Tx_0) - G(x_0)\|} \right]$$

$$+\gamma \left[ \frac{\|G(x_0) - T(x_0)\| + \|G(Tx_0) - T(Tx_0)\|}{2} \right] + \delta \left[ \frac{\|G(x_0) - T(Tx_0)\| + \|G(Tx_0) - T(x_0)\|}{2} \right]$$

$$+\eta \|G(x_0) - G(Tx_0)\| + \xi \left[ \frac{\|G(x_0) - T(x_0)\| \|G(Tx_0) - T(Tx_0)\|}{\|G(x_0) - G(Tx_0)\|} \right]$$

Putting  $Gx_0 = Tx_0$ ,

$$\|T(x_0) - x_0\| \geq \alpha \left[ \frac{\|T(x_0) - T(x_0)\|^2 \|x_0 - x_0\| + \|T(x_0) - T(x_0)\| \|x_0 - T(x_0)\|^2}{2 \|T(x_0) - T(x_0)\| \|x_0 - x_0\|} \right]$$

$$+\beta \left[ \frac{\|T(x_0) - T(x_0)\|^2 \|T(x_0) - x_0\| + \|x_0 - T(x_0)\|^2 \|T(x_0) - T(x_0)\|}{\|T(x_0) - T(x_0)\| \|x_0 - x_0\| + \|T(x_0) - T(x_0)\| \|x_0 - T(x_0)\|} \right]$$

$$+\gamma \left[ \frac{\|T(x_0) - T(x_0)\| + \|x_0 - x_0\|}{2} \right] + \delta \left[ \frac{\|T(x_0) - x_0\| + \|x_0 - T(x_0)\|}{2} \right] + \eta \|T(x_0) - x_0\|$$

$$+\xi \left[ \frac{\|T(x_0) - T(x_0)\| \|x_0 - x_0\|}{\|T(x_0) - x_0\|} \right]$$

$$\|T(x_0) - x_0\| \geq (\delta + \eta) \|T(x_0) - x_0\|$$

So  $Tx_0 = x_0$  ( $\delta + \eta > 1$ ) that is  $x_0$  is the fixed point of  $T$ . But  $Tx_0 = Gx_0$  so  $Gx_0 = x_0$ . Hence  $x_0$  is the fixed point of  $T$  and  $G$ .

Similarly above theorem the uniqueness part is obvious.

**Theorem 2.3:** Let  $T$  and  $G$  be two non expansion mapping of a Banach spaces  $X$  into itself. If  $T$  and  $G$  satisfies the following conditions:

1.  $T^2 = TG = G^2 = I$ , where  $I$  is identity mapping.
2.  $T$  and  $G$  commute.
- 3.

$$\|Tx - Ty\| \leq \alpha \left[ \frac{\|Gx - Tx\|^2 \|Gy - Ty\| + \|Gx - Tx\| \|Gy - Gx\|^2}{2 \|Gx - Tx\| \|Gy - Ty\|} \right]$$

$$+\beta \left[ \frac{\|Gx - Tx\|^2 \|Gx - Ty\| + \|Gy - Gx\|^2 \|Gx - Tx\|}{\|Gx - Tx\| \|Gy - Ty\| + \|Gx - Tx\| \|Gy - Gx\|} \right] + \gamma \left[ \frac{\|Gx - Tx\| + \|Gy - Ty\|}{2} \right]$$

$$+\delta \left[ \frac{\|Gx - Ty\| + \|Gy - Tx\|}{2} \right] + \eta \|Gx - Gy\| + \xi \left[ \frac{\|Gx - Tx\| \|Gy - Ty\|}{\|Gx - Gy\|} \right]$$

For every  $x, y \in X$  and  $\alpha, \beta, \gamma, \delta, \eta, \xi \geq 0$ . where  $20\alpha + 4\beta + 8\gamma + 8\delta + 4\eta + 16\xi < 8$ . Then  $T$  and  $G$  have common fixed point.

**Proof:** Suppose  $x$  is any point in Banach space. Taking  $y = \frac{1}{2} (T + I)x$ ,  $z = T(y)$ ,  $v = 2y - z$ , we have

$$\|z - x\| = \|Ty - T^2x\| = \|Ty - T(Tx)\|$$

$$\|z - x\| \leq \alpha \left[ \frac{\|y - Ty\|^2 \|Tx - T(Tx)\| + \|y - Ty\| \|Tx - y\|^2}{2 \|y - Ty\| \|Tx - T(Tx)\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \|y - T(Tx)\| + \|Tx - y\|^2 \|y - Ty\|}{\|y - Ty\| \|Tx - T(Tx)\| + \|y - Ty\| \|Tx - y\|} \right]$$

$$+\gamma \left[ \frac{\|y - Ty\| + \|Tx - T(Tx)\|}{2} \right] + \delta \left[ \frac{\|y - T(Tx)\| + \|Tx - Ty\|}{2} \right] + \eta \|y - Tx\| + \xi \left[ \frac{\|y - Ty\| \|Tx - T(Tx)\|}{\|y - Tx\|} \right]$$

$$\|z - x\| \leq \alpha \left[ \frac{\|y - Ty\|^2 \|Tx - x\| + \|y - Ty\| \|Tx - y\|^2}{2 \|y - Ty\| \|Tx - x\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \|y - x\| + \|Tx - y\|^2 \|y - Ty\|}{\|y - Ty\| \|Tx - x\| + \|y - Ty\| \|Tx - y\|} \right]$$

$$+\gamma \left[ \frac{\|y - Ty\| + \|Tx - x\|}{2} \right] + \delta \left[ \frac{\|y - x\| + \|Tx - y\| + \|y - Ty\|}{2} \right] + \eta \|y - Tx\| + \xi \left[ \frac{\|y - Ty\| \|Tx - x\|}{\|y - Tx\|} \right]$$

$$\|z - x\| \leq \alpha \left[ \frac{\|y - Ty\|^2 \|Tx - x\| + \|y - Ty\| \left\| Tx - \frac{1}{2} (T + I)x \right\|^2}{2 \|y - Ty\| \|Tx - x\|} \right]$$

$$+\beta \left[ \frac{\|y - Ty\|^2 \left\| \frac{1}{2} (T + I)x - x \right\| + \left\| Tx - \frac{1}{2} (T + I)x \right\|^2 \|y - Ty\|}{\|y - Ty\| \|Tx - x\| + \|y - Ty\| \left\| Tx - \frac{1}{2} (T + I)x \right\|} \right]$$

$$+\gamma \left[ \frac{\|y - Ty\| + \|Tx - x\|}{2} \right] + \delta \left[ \frac{\left\| \frac{1}{2} (T + I)x - x \right\| + \left\| Tx - \frac{1}{2} (T + I)x \right\| + \|y - Ty\|}{2} \right]$$

$$\begin{aligned}
 & +\eta \left\| \frac{1}{2}(T+I)x - Tx \right\| + \xi \left[ \frac{\|y - Ty\| \|Tx - x\|}{\left\| \frac{1}{2}(T+I)x - Tx \right\|} \right] \\
 \|z - x\| \leq & \alpha \left[ \frac{\|y - Ty\|^2 \|Tx - x\| + \|y - Ty\| \frac{1}{4} \|Tx - x\|^2}{2\|y - Ty\| \|Tx - x\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \frac{1}{2} \|Tx - x\| + \frac{1}{4} \|Tx - x\|^2 \|y - Ty\|}{\|y - Ty\| \|Tx - x\| + \|y - Ty\| \frac{1}{2} \|Tx - x\|} \right] \\
 & +\gamma \left[ \frac{\|y - Ty\| + \|Tx - x\|}{2} \right] + \delta \left[ \frac{\frac{1}{2} \|Tx - x\| + \frac{1}{2} \|Tx - x\| + \|y - Ty\|}{2} \right] + \eta \frac{1}{2} \|x - Tx\| + \xi \left[ \frac{\|y - Ty\| \|Tx - x\|}{\frac{1}{2} \|x - Tx\|} \right] \\
 \|z - x\| \leq & \alpha \left[ \frac{\|y - Ty\| \|Tx - x\| (\|y - Ty\| + \frac{1}{4} \|Tx - x\|)}{2\|y - Ty\| \|Tx - x\|} \right] + \beta \left[ \frac{\|y - Ty\| \|Tx - x\| (\frac{1}{2} \|y - Ty\| + \frac{1}{4} \|x - Tx\|)}{\frac{3}{2} \|y - Ty\| \|Tx - x\|} \right] \\
 & +\gamma \left[ \frac{\|y - Ty\| + \|Tx - x\|}{2} \right] + 2\xi \|y - Ty\| + \delta \left[ \frac{\|Tx - x\| + \|y - Ty\|}{2} \right] + \frac{\eta}{2} \|x - Tx\| \\
 \|z - x\| \geq & \frac{\alpha}{2} \|y - Ty\| + \frac{\alpha}{8} \|Tx - x\| + \frac{\beta}{3} \|y - Ty\| + \frac{\beta}{6} \|Tx - x\| + \frac{\gamma}{2} \|y - Ty\| + \frac{\gamma}{2} \|Tx - x\| + \frac{\delta}{2} \|Tx - x\| + \\
 & \frac{\delta}{2} \|Ty - y\| + \frac{\eta}{2} \|Tx - x\| + 2\xi \|y - Ty\| \\
 \|z - x\| \leq & \left( \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi \right) \|y - Ty\| + \left( \frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2} \right) \|Tx - x\| \tag{2.5} \\
 \|v - x\| = & \|2y - z - x\| = \left\| 2 \frac{1}{2} (T+I)x - Ty - x \right\| = \|Tx - Ty\| \\
 \|v - x\| \leq & \alpha \left[ \frac{\|y - Ty\|^2 \|x - Tx\| + \|y - Ty\| \|x - y\|^2}{2\|y - Ty\| \|x - Tx\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \|y - Tx\| + \|x - y\|^2 \|y - Ty\|}{\|y - Ty\| \|x - Tx\| + \|y - Ty\| \|x - y\|} \right] + \\
 & \gamma \left[ \frac{\|y - Ty\| + \|x - Tx\|}{2} \right] + \delta \left[ \frac{\|y - Tx\| + \|x - Ty\|}{2} \right] + \eta \|y - x\| + \xi \left[ \frac{\|y - Ty\| \|x - Tx\|}{\|y - x\|} \right] \\
 \|v - x\| \leq & \alpha \left[ \frac{\|y - Ty\|^2 \|x - Tx\| + \|y - Ty\| \left\| x - \frac{1}{2}(T+I)x \right\|^2}{2\|y - Ty\| \|x - Tx\|} \right] \\
 & +\beta \left[ \frac{\|y - Ty\|^2 \left\| \frac{1}{2}(T+I)x - Tx \right\| + \left\| x - \frac{1}{2}(T+I)x \right\|^2 \|y - Ty\|}{\|y - Ty\| \|x - Tx\| + \|y - Ty\| \left\| x - \frac{1}{2}(T+I)x \right\|} \right] \\
 & +\gamma \left[ \frac{\|y - Ty\| + \|x - Tx\|}{2} \right] + \delta \left[ \frac{\left\| \frac{1}{2}(T+I)x - Tx \right\| + \left\| x - \frac{1}{2}(T+I)x \right\| + \|y - Ty\|}{2} \right] \\
 & +\eta \left\| \frac{1}{2}(T+I)x - x \right\| + \xi \left[ \frac{\|y - Ty\| \|x - Tx\|}{\left\| \frac{1}{2}(T+I)x - x \right\|} \right] \\
 \|v - x\| \leq & \alpha \left[ \frac{\|y - Ty\|^2 \|x - Tx\| + \|y - Ty\| \frac{1}{4} \|x - Tx\|^2}{2\|y - Ty\| \|x - Tx\|} \right] + \beta \left[ \frac{\|y - Ty\|^2 \frac{1}{2} \|x - Tx\| + \frac{1}{4} \|x - Tx\|^2 \|y - Ty\|}{\|y - Ty\| \|x - Tx\| + \|y - Ty\| \frac{1}{2} \|x - Tx\|} \right] \\
 & +\gamma \left[ \frac{\|y - Ty\| + \|x - Tx\|}{2} \right] + \delta \left[ \frac{\frac{1}{2} \|x - Tx\| + \frac{1}{2} \|x - Tx\| + \|y - Ty\|}{2} \right] + \eta \frac{1}{2} \|Tx - x\| + \xi \left[ \frac{\|y - Ty\| \|x - Tx\|}{\frac{1}{2} \|Tx - x\|} \right] \\
 \|v - x\| \leq & \alpha \left[ \frac{\|y - Ty\| \|x - Tx\| (\|y - Ty\| + \frac{1}{4} \|x - Tx\|)}{2\|y - Ty\| \|x - Tx\|} \right] + \beta \left[ \frac{\|y - Ty\| \|x - Tx\| (\frac{1}{2} \|y - Ty\| + \frac{1}{4} \|x - Tx\|)}{\frac{3}{2} \|y - Ty\| \|x - Tx\|} \right] + \\
 & \gamma \left[ \frac{\|y - Ty\| + \|x - Tx\|}{2} \right] + \delta \left[ \frac{\|x - Tx\| + \|y - Ty\|}{2} \right] + \frac{\eta}{2} \|Tx - x\| + 2\xi \|y - Ty\| \\
 \|v - x\| \leq & \frac{\alpha}{2} \|y - Ty\| + \frac{\alpha}{8} \|x - Tx\| + \frac{\beta}{3} \|y - Ty\| + \frac{\beta}{6} \|x - Tx\| + \frac{\gamma}{2} \|y - Ty\| + \frac{\gamma}{2} \|x - Tx\| + \frac{\delta}{2} \|x - Tx\|
 \end{aligned}$$

$$\begin{aligned} & + \frac{\delta}{2} \|Ty - y\| + \frac{\eta}{2} \|x - Tx\| + 2\xi \|y - Ty\| \\ \|v - x\| & \leq \left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right) \|y - Ty\| + \left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right) \|x - Tx\| \end{aligned} \quad (2.6)$$

Now, by (2.5) and (2.6), we get

$$\begin{aligned} \|z - v\| & \leq \|z - x\| + \|x - v\| \\ \|z - v\| & \leq 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right) \|y - Ty\| + 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right) \|x - Tx\| \end{aligned} \quad (2.7)$$

Also,

$$\|z - v\| = \|Ty - (2y - z)\| = \|Ty - 2y + Ty\| = 2 \|Ty - y, u\| \quad (2.8)$$

Therefore by (2.7) and (2.8), we have

$$2 \|Ty - y, u\| \leq 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right) \|y - Ty\| + 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right) \|x - Tx\|$$

$$2 \|Ty - y, u\| - 2\left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right) \|y - Ty\| \leq 2\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right) \|x - Tx\|$$

$$\|x - Tx\| \geq \frac{1 - \left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)}{\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)} \|y - Ty\|$$

$$\|x - Tx\| \geq m \|y - Ty\| \quad \text{As } 20\alpha + 4\beta + 8\gamma + 8\delta + 4\eta + 16\xi < 8.$$

$$\text{where } m = \frac{1 - \left(\frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{\delta}{2} + 2\xi\right)}{\left(\frac{\alpha}{8} + \frac{\beta}{6} + \frac{\gamma}{2} + \frac{\delta}{2} + \frac{\eta}{2}\right)} > 1$$

Let  $R = \frac{1}{2} (T + I)$  then for every  $x \in X$ ,

$$\|R^2x - Rx\| = \|R(Rx) - Rx\| = \|Ry - y\| = \frac{1}{2} \|y - Ty\| = \frac{1}{2} \|T(Ty) - Ty\| = \frac{1}{2} \|Ty - y\|$$

$$\Rightarrow \frac{1}{2} \|y - Ty\| \Rightarrow \frac{1}{2} \|TGy - Ty\| \leq \frac{1}{2} \|Gy - y\|$$

Because  $T$  is non expansive. so  $\|R^2(x) - R(x)\| \leq \frac{m}{2} \|x - Tx\|$

$\|R^2x - Rx\| < \frac{m}{2} \|x - Tx\|$  By the definition of  $m$ . we claim that  $R^n(x)$  is a cauchy sequence in  $X$ . Also by completeness  $R^n(x)$  converges to  $R(x)$ , i.e.,  $\lim_{n \rightarrow \infty} R^n(x) = x_0 \Rightarrow R(x_0) = x_0$ . therefore  $x_0$  is fixed point of  $T$ .

$$\begin{aligned} \text{Again, } \|R^2(x) - R(x)\| & \leq \frac{m}{2} \|x - Tx\| = \frac{m}{2} \|G^2x - G^2Tx\| = \frac{m}{2} \|G(Gx) - GT(Gx)\| \\ & \leq \frac{m}{2} \|Gx - x\| \quad \text{Because } G \text{ is non-expansive.} \end{aligned}$$

We can conclude that  $G(x_0) = x_0$  that is  $x_0$  is fixed point of  $G$ . Therefore  $T(x_0) = G(x_0) = x_0$ . So  $x_0$  is a common fixed point of  $T$  and  $G$ .

The uniqueness part is obvious.

**III. Conclusion:** In this paper we proved fixed point theorem and common fixed point theorem in 2-Banach space. The results of this paper extend the previously known results Banach space and 2-Banach space.

**Acknowledgement:** The Authors are thankful to the anonymous referees for his valuable suggestions for the improvement of this paper.

## References:

1. Browder, F.E. "Non-expansive non-linear operators in Banach space". Proc. Nat. Acad. Sci. U.S.A. vol-54, pp-1041-1044, 1965.
2. Emmanuele, G., "Fixed point theorems in complete metric space". Not linear Anal. vol 5, pp-287-292, 1981.
3. Gohde Zum prizip dev kontraktiven abbildung. Math.Nachr. 30, 251-258, 1965.
4. Dr.K.V.Radha, Mass Transfer Correlation for Decolorization Studies in Immobilized Packed Bed Bioreactor; Engineering Sciences international Research Journal: ISSN 2320-4338 Volume 3 Issue 1 (2015), Pg 17-19
5. Goebel, K., "An elementary proof of the fixed theorem of Browder and Kirk". Michigan Math. J. vol 16, pp-381-383, 1969.
6. Goebel, K., and Zlotkiewicz, E. "Some fixed point theorems in Banach spaces". Collq. Math. Vol 23, pp-103-106, 1971.
7. Iseki, K., "Fixed point theorem in Banach spaces", Math. Sem. Notes Kobe Univ. vol.2(1), pp3,4, 1974.
8. M. Venkateswarlu, B. Rajini Kanth, A Simple Method for the Determination of Magneto Rheological Properties of Nanoparticulated Cobalt Ferrite Based Magnetorheological Fluids; Engineering Sciences international Research

- Journal: ISSN 2320-4338 Volume 3 Issue 1 (2015), Pg 20-23
9. Isekey, K., Sharma, P.L. and Rajput, S.S. "An extension of Banach contraction principle through rational expression" Mathematics seminar notes Kobe University vol 10, pp-677-679 ,1982.
  10. Singh, M.R. and Chatterjee. "Fixed point theorems in Banach space". Pure Math. Manu.vol.6, pp53-61,1987.
  11. Shams-ur-rahman, Beg, I. and Tuladhar, B.M. "Approximating Fixed Points of Non-self Nonexpansive Mappings in Fuzzy Normed Spaces", Int. Journal of Math. Analysis, Vol. 7, no. 27, pp-1341 - 1346, 2013.
  12. Arnab Acharyya, Application of Image Halftone Technique in Visual Secret Sharing of Text Image; Engineering Sciences international Research Journal: ISSN 2320-4338 Volume 3 Issue 1 (2015), Pg 24-27
  13. Sharma, PL and Rajput, SS. "Fixed point theorems in Banach space". Vikram math. Jour. vol.4, 35,1983.
  14. Ar. Arpan Dasgupta, Dr. Madhumita Roy, Comparing Visual Comfort in Daylight Pattern Between An Old and A Modern office Building; Engineering Sciences international Research Journal: ISSN 2320-4338 Volume 3 Issue 1 (2015), Pg 28-35
  15. Kirk, W.A. "Fixed point theorem for non-expansive mapping-II contemp". Math. Vol 18, pp-121-140, 1983.
  16. Datson, W.G.Jr., "Fixed points of quasi non-expansive mappings". J.Austral. Math. Soc.vol 13, pp167-170 ,1972.
  17. Dhaneswar Kalita, Sub-Linear Dc Photoconductivity of thermally Evaporated Polycrystalline CDSE Thin Films; Engineering Sciences international Research Journal: ISSN 2320-4338 Volume 3 Issue 1 (2015), Pg 36-44

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