A BOUNDARY VALUE PROBLEM OF FRACTIONAL ORDER OF EXISTENCE OF SOLUTIONS THE HALF - LINE VIA MONOTONE THEORY

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Abstract: A boundary value problem we discuss existence and uniqueness of a weak solution of a fractional problem on the half-line via the Minty - Browder theorem.

Keywords: Monotone operatior, hemicontinuours operator, demicontinuous operator. Minty - Browder theorem, fractional B.V.P.s weak solution, uniqueness.

Introduction: Fractional Calculus is a generalization of ordinary differentiation and integration to an arbitvary order. In this paper we study to fractional boundary value problem.

$$\boldsymbol{D^\propto - (D^\propto + u(t)) + u(t) = ((t_1 u(t)), t \in (0,+\infty)}$$

$$\boldsymbol{u(0) = u(+\infty) = 0}$$

Where $\frac{1}{2} <\propto 1$ *and* $f; (0+\infty)\times\mathbb{R}\to\mathbb{R}$ **is a caratheodory function first we recall fractionalintegral and derivatives operators:**

Definition 1.1. ([5], [8], [7]) Let μ ba a function defined on (0, +∞). The left and right Riemann – Liouville fractional integrals of order α > 0 for a function μ denoted by $I^\alpha + \mu$ and $I^\alpha - \mu$, respectively, are defined by

$$I^\alpha + \mu\ (t) = \frac{1}{I\ (\alpha)}\int_0^t (t-s)^{\alpha-1}\mu\ (s)\ ds,\ t \in (0, +\infty),$$

And

$$I^\alpha + \mu\ (t) = \frac{1}{I\ (\alpha)}\int_t^{+\infty} (s-t)^{\alpha-1}\mu\ (s)\ ds,\ t \in (0, +\infty),$$

Provided that the right – hand side is pointwise defined on (0, + ∞); here T (α) is the gamma function.

Definition 1.2: ([5], [8], [7]) Let μ be a function defined on (0, +∞). For N – 1 ≤ α < n (n ∈ N*), the left and right Riemann – Liouville fractional derivatives of order α for a function μ denoted by $D^\propto + \mu$ and $D^\propto - \mu$ respectively, are defined by

$$D^\propto + \mu(t) = \frac{d^n}{dt^n} I^{n-\alpha} + u(t) = \frac{1}{T\ (n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\propto-1}\mu(s)ds,\ t \in (0,+\infty),$$

And

$$D^\propto - u(t) = (-1)^n \frac{d^n}{dt^n} I^{n-\propto} u(t) = \frac{(-1)^n}{T(n-\propto)}\frac{d^n}{dt^n}\int_t^{+\infty}(s-t)^{n-\propto-1} u\ (s)\ ds,\ t \in (0, +\infty),$$

Provided that the right – hand side is pointwise defined.

In particular for α=n, $D^\propto + u\ (t) = D^\propto u(t) = D^n u(t)$ *and* $D^\propto - u(t) = (-1)^n D^n u(t),\ t \in (0,+\infty)$

Proposition 1.1.[5] If $D^\propto + u\ (t) = D^\propto - u \in L^1(0,+\infty)$ *and* $n-1 \le \propto < n$, *then*

$$I^\propto + D^\propto + u(t) = u(t) + \sum_{j=1}^n Cj\ (t-a)^{\propto - j}$$

$$I^\propto - D^\propto - u(t) = u(t) + \sum_{j=1}^n Cj\ (b-t)^{\propto - j}$$

With $C_j^1 + \frac{(-1)^{\propto-1} D_{b-}^{\propto-1}}{T(\propto-j+1} \in R,\ j = 1,2 \ldots, n.$

Now we introduce a new space which is suitable for the study of our fractional BVP. Let.

$$E_0^\propto(0,+\infty) = \{u \in L^2(0,+\infty).\ D^\propto + u \in L^2(0,+\infty), u(0) = u(\infty) = 0\},$$

With the natural norm

$$\|u\|\propto = (\int_0^{+\infty}|u(t)|^2 dt + \int_0^{+\infty}|D^a + u(t)|^2 dt)^{\frac{1}{2}}, \forall u \in E_0^\propto(0,+\infty). \quad (1.2)$$

Let the space $C_p([0,+\infty))$ *be defined by*

$$C_p([0,+\infty)) = \{u \in C([0,+\infty)), R): \lim p(t)u(t) exists\}$$

And endowed with the norm

$$||u||\infty, p = \sup_{t\in[0,+\infty)} p(t)|u(t)|,$$

Where the function p :[0, +∞) → (0,+∞) is continuous and satisfies

$$\lim_{n\to+\infty} p(t)t^{\propto-\frac{1}{2}} = 0.$$

We put

$$M = \frac{1}{\sqrt{2\propto-1}.\, r(\propto)} \cdot \sup_{t>0} p(t)t^{\propto-\frac{1}{2}}.$$

Throughout this paper we assume p satisfies these conditions. Using the same idea as in [5] one can casily prove the following proposition.

Proposition 1.2: [5] If u∈ $L^2(0,+\infty), D^\propto + u \in L^2(0,+\infty)$ *with* $u(0) = U(+\infty) = 0$ *and* $u \in C_0^\infty([0,+\infty))$, *then*

$$\int_0^{+\infty} D^\propto + u(t)u(t)dt = \int_0^{+\infty} u(t)D^\propto - u(t)dt.$$

Using Proposition 1.2. wo now define a weak solution of problem (1.1).

Definition 1.3.: A weak solution of the fractional boundary value problem (1.1) is given by a solution of the following variational formula.

$$\int_0^{+\infty}\left[D^{\propto}+u(t)D^{\propto}+u(t)+u(t)u(t)-f\big(t,u(t)\big)u(t)\right]dt=0, for\ all\ u\in E_0^{\propto}(0,+\propto).$$

Now we recall some information for the literature needed in this paper.

Definition 1.4.: [9] Let X be a Banach space. An operator A : X→ X^* which satisfies.

$$\langle Au-Au,u-u\rangle\ \geq 0$$

For any u, $\nu\in X$ is called a monotone operator. An operator A is called strictly monotone it for u $\neq\nu$ striet inequality holds in (1.3). An operatior A is called strongly monotone if there exists C >0 such that.

$$\langle Au-Au,u-u\rangle\ \geq C||u-u||^2$$

For any u, $\nu\ \in X$. It is clear that a strongly monotone is strictly monotone.

Detinition 1.5. [9] Let A : X → X^* be an operator on the real Banach
space X.

(a) A is said to be demicontinuous if

$u_n\to u\ as\ n\to\ +\infty\ implies\ Au_n\to Au\ as\ n\to+\infty.$

(b) A is said to be hemicontionus if the real function. t→ $\langle A(u+tu),w\rangle$ is continuous on [0.1] for al u, v, $w\in X.$

(c) A is said to be coercive if

$$||u||^{lim}\to+\infty\ \frac{\langle Au.u\rangle}{||u||}=\ +\infty.$$

Remark 1.1. [4] It is easy to see that for monotone operator A : X→ X^*
With Dom (A) = X, demicontinuity and hemicontinuity are equivalent.

Theorem 1.3. [6] (Minty-Browder) Let X be a reflexive real Banach space.
Let A : X → X^* be an operator which is bounded, hemicontious, coercive and monotone on the space X. Then, the equation Au = f has at least one solution u$\in X$ for each f$\in X^*$. If a is strictly monotone then the solution is unique.

2. Main Result:

We begin with the space $E_0^{\propto}(0,+\infty)$.

Proposition 2.1. $E_0^{\propto}(0,+\infty)$. is a Banach space.

Proof. Let $(u_n)\ n\geq 1$ be a Causchy sequence in $E_0^{\propto}(0,+\infty)$. Then $(u_n)\ n\geq 1$, $(D^{\propto}+un)n\geq 1$ are Cauchy sequences in $L^2(0,+\infty)$. From (1.2) we have $||u_n-u_m||_{\propto}\to 0\ as\ n,m\to+\infty$ which implies that

$$||u_n-u_{m||L^2\to 0,||D^{\propto}+u_n-D^{\propto}+u_m||L_2\to 0}$$

As n, m→ $+\infty. Since\ L^2(0+\infty)$ is a Banach space, there exist functions $u_{1,}u_2\in\ L^2(0+\infty)$ such that $u_1\to u_1, D^{\propto}+u_n\to u_2\ in\ L^2(0,+\infty) as$
n→ +∞. We now show that $D^{\propto}+u_1=u_{2.}$ From Proposition 1.2, we have

$$\int_0^{+\infty} D^{\propto}+u_n(t)dt=\int_0^{+\infty}u_n(t)D^{\propto}-\varphi(t)dt,\forall\varphi\in C_0^{\infty}([0,+\infty))$$

And then by using the definition of the inner product in $L^2(0,+\infty)$, we obtain that

$$\int_0^{+\infty} u_2(t)dt=\int_0^{+\infty}u_1(t)D^{\propto}-\varphi(t)dt,\forall\varphi\in C_0^{\infty}([0,+\infty))$$

And so $D^{\propto}+u_1. Thus\ \lim_{n\to+\infty}\left||u_n-u_1\right||\propto=0, and\ so\ E_0^{\propto}(0,+\infty)$ is a Banach space.

Lemma 2.2. The operator
T : $E_0^{\propto}(0,+\infty)\to T\left(E_0^{\propto}(0,+\infty)\right)\subset\ L^2(0,+\infty)=L_2^2(0,+\infty)$

$$u\to T(u)=(u,D^{\propto}+u)$$

Is an isometric isomorphic mapping.

Proof. It is clear that T is a linear operatior and we now show that T conserves norms, i.e.

$$\forall u\in\ E_0^{\propto}(0,+\infty):\left||Tu\right||L_2^2=\left||u\right||\propto.$$

Indeed, we have

$$||(u,D^{\propto}+u)\,||L_2^2=||u||\propto$$

$$\Leftrightarrow\left||u\right||L_2+\left||D^{\propto}+u\right||L^2=\left||u\right||\propto.$$

■

Proposition 2.3. $E_0^{\propto}(0,+\infty)$ is a reflexive space.

Proof. Since,$L^2((0,+\infty))\mathbb{R})$ is a reflexive Banach space, the Cartesian space

$$L_2^2(0,+\infty)\mathbb{R})=L^2((0,+\infty)\mathbb{R})\times L^2(0,+\infty)\mathbb{R})$$

Is also a reflexive Banach space with respect to the norm.

$$||u||L_2^2=\sum_{u=1}^2\left||u_i\right||L^2 where\ u=(u_1,u_2)\in L_2^2(0,+\infty)\mathbb{R}).$$

Then

T : $E_0^{\propto}(0,+\infty)\to T(E_0^{\propto}(0,+\infty)\subset L_2^2(0,+\infty)$

u→ $T(u)=(u,D^{\propto}+u)$

is an isometric isomorphic. So T $(E_0^{\propto}(0,+\infty))$ is a closed subspace of $L_2^2(0,+\infty)$
and by [[2], Theorem 4.10.5] then T($E_0^{\propto}(0,+\infty)$) is reflexive. Consequently
$E_0^{\propto}(0,+\infty)$ is also reflexive (see[[2], Lemma 4.10.4]).

Proposition 2.4$E_0^{\propto}(0,+\infty)$ is a separable space.

Proof. Since, L^2 $(0,+\infty),\mathbb{R})$ is a separable Banach space, the Cartesian space

$$L_2^2(0,+\infty)\mathbb{R})=L^2\ (0,+\infty)\mathbb{R})\times L^2\ (0,+\infty)\mathbb{R})$$

Is also a separable Banach space with respect to the norm.

$||u||L_2^2\sum_{i=1}^2\left||u_i\right||L^2 where\ u=(u_1,u_2)\in L_2^2(0,+\infty)\mathbb{R}).$

Then, the space T($E_0^{\propto}(0,+\infty)$) $\subset\ L_2^2$ is also separable (see [1], Proposition
111.22). Morcover, the operaor

T : $E_0^{\propto}(0,+\infty)\to T(E_0^{\propto}(0,+\infty)\subset L_2^2(0,+\infty)$

u→ $T(u)=(u,D^{\propto}+u)$

is an isometric isomorphic, so $E_0^{\propto}(0,+\infty)$ is a separable space.

Lemma 2.5. For all u∈ $E_0^{\propto}(0,+\infty)$ we have that $E_0^{\propto}(0,+\infty)$ embeds continuously in $C_p\ (0,+\infty))$. i.e.,

$$\exists M_0>0, \left|\left|u\right|\right|_{\infty,p}\le M_0\left|\left|u\right|\right|_{\propto}.$$

Proof. For all u∈ $E_0^{\propto}(0,+\infty), and\ t>0,$

U(t)=$I_+^{\propto}\left(D^{\propto}+u(t)\right),$

So

$$P(t)u(t)=p(t)\ I_+^{\propto}\left(D^{\propto}+u(t)\right)$$

Which implies from the Cauchy-Schwartz inequality

$$\left|p(t)I_+^{\propto}\left(D^{\propto}+u(t)\right)\right|$$

$$=\frac{p(t)}{T(\propto)}\left|\int_0^t(t-s)^{\propto-1}D^{\propto}+u(s)ds\right|$$

$$\le\frac{p(t)}{T(\propto)}\left(\int_0^t(t-s)^{2(\propto-1)}ds\right)^{\frac12}\left(\int_0^t D^{\propto}+u(s))^2ds\right)^{\frac12}$$

$$\le\frac{p(t)}{T(\propto)}\left(\int_0^t(t-s)^{2(\propto-1)}ds\right)^{\frac12}\left(\int_0^{+\infty}|u(s)|^2ds+\int_0^{+\infty}|D^{\propto}+u(s)|^2ds\right)^{\frac12}$$

$$=\frac{\left|\left|u\right|\right|\propto}{\sqrt{2\propto-1.T(\propto)}}p(t)t^{\propto-\frac12}$$

Then

$$||u||\infty,p=\sup_{t\in[0,+\infty)}|p(t)u(t)|$$

$$=\sup_{t\in[0,+\infty)}\left|p(t)I_+^{\propto}\left(D^{\propto}+u(t)\right)\right|$$

$$\le\frac{||u||\propto}{\sqrt{2\propto-1.T(\propto)}}.\sup_{t>0}p(t)t^{\propto-\frac12},$$

And so,

$$||u||\infty,p\le M\left|\left|u\right|\right|\propto.$$

From the definition of the norm in $E_0^{\propto}(0,+\infty)$, it is easy to see that ■

Proposition 2.6. $E_0^{\propto}(0,+\infty)$ embeds continuously in $L^2\ (0,+\infty)$.

To prove the compactness embedding of $E_0^{\propto}(0,+\infty)\ in\ C_p\left([0,+\infty)\right)$ we follow the ideas in [3].

Lemma 2.7.[3] I.ct D⊂ $C_p([0,+\infty))$ be a bounded set. Then D is relatively compact if the following conditions hold.

(a) D is equicontinuous on any compact sub-interval of $\mathbb{R}^+, i.e.,$

$$\forall J\subset[0,+\infty) compact\ subinterval,\forall\in>0,\exists\delta>0,\forall t_1,t_2\in J,$$

$$|t_1-t_2|<\delta\Rightarrow|p(t_1)u(t_1)-p(t_2)u(t_2)|\le\varepsilon,\forall u\in D,$$

(b) Dis equiconvergent at +∞ i.e.,

$$\forall\varepsilon>0,\exists\ T=T(\varepsilon)>0 such\ that$$

$$\forall t_1,t_2:\ t_1,t_2\ge T(\varepsilon)\Rightarrow|p(t_1)u(t_1)-|p(t_2)u(t_2)|\le\varepsilon,\forall u\in D.$$ ■

Theorem 2.8.The embedding

$$E_0^{\propto}(0,+\infty)\to C_p([0,+\infty))$$

Is compact.

Proof. Let D ⊂ $E_0^{\propto}(0,+\infty)$ be a bounded set. Then it is bounded in $C_p([0,+\infty))$ by Lemma 2.5. Let R> 0 be such that for all $u\in D\left|\left|u\right|\right|\propto\le$ R.

We will apply Lemma 2.7.

(a) D is equicontinuous on every compact interval of $[0,+\infty)$.

Let $u\in D\ and\ t_1,t_2\in J\subset[0,+\infty)$. where J is a compact sub-interval and by the Cauchy-Schwarz inequality, we have

$$|p(t)I^{\propto}+u(t1)-p(t_2)I^{\propto}+u(t_2)=$$

$$\frac{1}{T(\propto)}\left|\ p(t_1)\int_0^{t_1}(t_1-s)^{\propto-1}u(s)ds\right.$$

$$-\quad p(t_2)\int_0^{t_1}(t_1-s)^{\propto-1}u(s)ds\ |$$

$$\le\frac{1}{T(\propto)}\left|\ p(t_1)\int_0^{t_1}(t_1-s)^{\propto-1}u(s)ds\right.$$

$$-\quad p(t_2)\int_0^{t_1}(t_1-s)^{\propto-1}u(s)ds\ |$$

$$+\ \frac{p(t_2)}{T(\propto)}\left|\int_{t_1}^{t_2}(t_2-s)^{\propto-1}u(s)ds\right|$$

$$\le\frac{1}{T(\propto)}\int_0^{t_1}|\ p(t_1)\ (t_1-s)^{\propto-1}p(t_2)(t_2-s)^{\propto-1})||u(s)|ds$$

$$+\frac{p(t_2)}{T(\propto)}\left|\int_{t_1}^{t_2}(t_2-s)^{\propto-1}u(s)ds\right|$$

$$\le\frac{||u||L^2}{T(\propto)}\left[\left(\int_0^{t_1}(\ p(t_2)(t_1-s)^{\propto-1}-p(t_2)(t_2-s)^{\propto-1})^2ds\right)^{1/2}\right.$$

$$\left.-\quad p(t_2)\int_{t_1}^{t_2}(t_2-s)^{2\propto-2}ds\right]$$

So we have

$$|p(t_1)u(t_1)-\ |p(t_2)u(t_2)|$$

$$=p(t_1)I^{\propto}+D^{\propto}+u(t_1)-\ p(t_2)I_0^{\propto}+D^{\propto}+u(t_2)|$$

$$<\frac{||D^{\propto}+u||L^2}{T(\propto)}\left(\int_0^{t_1}\left(\ p(t_1)(t_1-s)^{\propto-1}-\ p(t_2)(t_2-s)^{\propto-1}\right)^2ds)^{1/2}\right)$$

$$+\frac{||D^{\propto}+u||L^2}{T(\propto)}\ p(t_2)\left(\int_{t_1}^{t_2}(t_2-s)^{2\propto-2}ds\right)^{1/2}$$

$$\le\frac{R}{T(\propto)}\left(\int_0^{t_1}\left(\ p(t_1)(t_1-s)^{\propto-1}-\ p(t_2)(t_2-s)^{\propto-1}\right)^2ds)^{1/2}\right)$$

$$+\frac{R}{T(\propto)}\ p(t_2)\left(\int_{t_1}^{t_2}(t_2-s)^{(\propto-1)}ds\right)^{\frac12}\ \to\ 0.$$

As $|t_1-t_2|\ \to0.$

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