# FUZZY FUNCTIONAL ANALYTIC STUDY OF STATISTICAL CONVERGENCE IN VIEW OF FUZZY SEQUENCE SPACE

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**Abstract.** The aim of this work is to extend and study the idea of fuzzy statistical convergence, fuzzy lacunary convergence and fuzzy lacunary statistical convergence for the fuzzy sequence space $F(L(\mathbb{R}), M, p, s)$ of the real valued fuzzy number sequences by utilizing the modern techniques of fuzzy functional analysis. This sequence space is an unified approach of the class of sequences of fuzzy numbers $L(\mathbb{R})$, where an Orlicz function $M$, a normal sequence space $F$, a bounded sequence of strictly +ve real numbers $p$ ={$p_k$} with inf $p_k$ ≥0 and a multiplier sequence $(k^{-s})$, s ≥ 0 have been used. Further, some important results regarding these type of convergences and their inter-relations have been attempted to derive which are analogous to the results provided by the classical analysis.

**Keywords:** fuzzy statistical convergence, fuzzy lacunary convergent space, fuzzy lacunary statistical convergence, para-normed space, normal space.
**AMS subject classification:** 40A05,40A35, 40C05, 40D05, 46A45.

**Introduction:** Fuzzy number sequence space is one of the active areas of research of fuzzy functional analysis. The concept of fuzzy sets and fuzzy sets operations were first introduced by Zadeh [17] and Matloka [18] first introduced bounded and convergent sequences of fuzzy numbers and studied their properties. Later on,applying the notion of fuzzy real numbers, fuzzy real-valued sequence space was first introduced and studied by Nanda [21]. He showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Later on the study of theory of fuzzy sequence spaces hasbeen continued by Nanda and Tripathy[22], Savas [9], Mursaleen et al..[19], Fang et al..[16], Talo et al.. [23, 24], Dutta et al.[13], Esi et al.[28], Tripathy et al [7, 8], Basu et al [1] and many others.

Nuray and Savas [11] first introduced the notion of statistically convergent and statistically Cauchy sequence forsequence of fuzzy numbers. Later on Nurray [12] studied a related concept, namely, lacunary convergent and lacunarystatistical convergent sequences of Fuzzy numbers and various inclusion relations. Recently the notion of Statisticallyconvergent, lacunary statistical convergent sequences of fuzzy numbers defined by modulus and Orlicz functionhave been studied by Savas ([10]), Dutta et al. [13], Esi et al [4], Sarma [5], Tripathy et al [6]and many authors. Later on Mursaleen and Basarir [19], Fang and Huang [29]and many other authors studied the different convergence methods for fuzzy single and doublesequences.

A sequence of positive integers $\theta = (k_r)$ is called 'lacunary' if $k_0 = 0,\ 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ are denoted by $I_r$, where $I_r = (k_{r-1},\ k_r]$and the ratio $\frac{k_r}{k_{r-1}}$is denoted by $q_r$.

The space of lacunary strong convergence $N_\theta$has been introduced by Freedman et al. [3]. Later on Nurray [16] studied arelated concept, namely, lacunary convergent and lacunary statistical convergent sequences of Fuzzy numbers and variousinclusion relations.

The present author has made an attempt to fuzzify the concepts of statistical and lacunary statistical convergencemethods on the generalized composite class $\mathrm{F}(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s})$of fuzzy Orlicz sequence spaces introduced by Basu et al. [2].

**The class $\mathbf{F}(\mathbf{L}(\mathbb{R}), \mathbf{M}, \mathbf{p}, \mathbf{s})$:** Let $(\mathrm{F}, g_F)$ be a normal sequence space with paranorm $g_F$ which satisfies the following properties:

i) $g_F$ is monotone and;

ii) co-ordinatewise convergence implies convergence in paranorm $g_F$ i.e., if $(X^n) = (X_k^n) \in \mathrm{F}$, then

$X_k^n \longrightarrow 0$ as $n \longrightarrow \infty \Longrightarrow g_F\,(X^n) \longrightarrow 0$ as $n \longrightarrow \infty$

Let $M$ be a Orlicz function. We define

$$\mathrm{F}(L(\mathbb{R}), M, p, s) = \left\{X = (X_k): X_k \in\ L(\mathbb{R}), \left(k^{-s}\left[M\left(\frac{\bar{d}(X_k,\ \overline{0})}{\rho}\right)\right]^{p_k}\right) \in F \text{ for some } \rho > 0\right\}$$

where {$p_k$} is a bounded sequence of strictly +ve real numbers with $inf p_k > 0$ and $s \geq 0$.

We define a metric topology on this fuzzy sequence space $\mathrm{F}(L(\mathbb{R}), M, p, s)$ as follows:

For $X = (X_k),\ Y = (Y_k) \in \mathrm{F}(L(\mathbb{R}), M, p, s)$,

$$d(X, Y) = inf\left\{\rho^{p_k/T} > 0: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(x_k, y_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\right\}$$

where $T = \max(1,\ H)$ , $H = \sup_k p_k < \infty, \inf_k\ p_k > 0$.

**Particular Cases :** It is shown that this generalized class of sequences of fuzzy real numbers gives rise to many well known fuzzy sequence spaces on specifying the sequence space $F$, the Orlicz function $M$, the bounded sequence $\{p_k\}$ of positive real numbers and $s \geq 0$ as follows:

1. If we take $F = c, \ell_p, s = 1, p_k = 1$, for each $k \in N, M(x) = x$, for all $x \in [0, \infty)$, $F(L(\mathbb{R}), M, p, s)$ reduces to the spaces of Nanda [21];
2. If we take $F = w(p), c_0(p), s = 1, M(x) = x$, for all $x \in [0, \infty)$, $F(L(\mathbb{R}), M, p, s)$ reduces to the spaces of Mursaleen et al. [22];
3. If we take $F = \ell(p)[26], s = 1, M(x) = x$, for all $x \in [0, \infty)$, $F(L(\mathbb{R}), M, p, s)$ reduces to the spaces of Nurray and Savas [16];
4. If we take $F = \ell_\infty, c, c_0, \ell_p, M(x) = x$, for all $x \in [0, \infty), s = 1, p_k = 1$, for each $k \in N$, $F(L(\mathbb{R}), M, p, s)$ gives rise to the space of Talo et al. [23].

The following definitions and preliminary conceps have been used throughout the work

**Definitions and Propositions:**

**Definition 1. (Paranorm)[15]** Let $X$ be a linear space. Then $g: X \longrightarrow \mathbb{R}$ is called a paranorm on $X$ if for $x, y \in X$ and any scalar $\lambda$, (i) $g(x) \geq 0$; (ii) $x=\theta \Rightarrow g(x) = 0$; (iii) $g(x) = g(-x)$; (iv) $g(x+y) \leq g(x) + g(y)$; (v) $g(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$ whenever $\lambda_n \to \lambda$ and $x_n \to x$ for scalars $\lambda_n, \lambda$ and $x_n, x \in X$.

**Definition 2. (Monotone paranorm)[15]** A paranorm $g$ on a scalar valued paranormed sequence space $Z$ is called monotone paranorm if $X = (x_k) \in Z$, $Y = (y_k) \in Z$ and $|x_k| \leq |y_k|$ implies $g(x) \leq g(y)$. The space $(Z, g)$ is called monotone paranormed space.

**Definition 3.**[22] A sequence space $\lambda$ of sequences fuzzy numbers is said to be solid (or normal) if $(Y_n) \in \lambda$, whenever $\bar{d}(Y_n, \bar{0}) \leq \bar{d}(X_n, \bar{0})$ for some $(X_n) \in \lambda$.

**Definition 4.** [22] A metric $\bar{d}$ is said to be translation invariant metric on $L(\mathbb{R})$ if $\bar{d}(X+Z, Y+Z) = \bar{d}(X, Y)$ for X, Y, Z $\in L(\mathbb{R})$.

**Proposition 1.** [22] Let $X, Y, Z, W \in L(\mathbb{R})$ and $k \in \mathbb{N}$. Then,

(i) $(L(\mathbb{R}), \bar{d})$ is a complete metric space [20];

(ii) $\bar{d}(kX, kY) = |k|\, \bar{d}(X, Y)$

(iii) $\bar{d}(X+Y, W+Z) \leq \bar{d}(X, W) + \bar{d}(Y, Z)$;

**Proposition 2.** [22] Let $X, Y, Z, W \in L(\mathbb{R})$ and $k \in \mathbb{N}$. Then

(i) If $X_k \longrightarrow 0$ as $k \longrightarrow \infty$ then $\bar{d}((X_k, \bar{0}) \to \bar{d}((X, \bar{0})$ as $k \longrightarrow \infty$;

**Proposition 3.** [16] Let $(p_k)$ be a bounded sequence of strictly positive real numbers with $0 < p_k \leq \sup_k p_k = H, D = max(1, 2^{H-1}), T = max(1, H)$. Then

1. $|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}$;
2. $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$;

**Natural Density of a set:** The natural density of a set A is defined by

$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n: k \in A\}|$, where

the vertical bar denotes the cardinality of the set enclosed.

**Orlic Function: [20]]:** An Orlicz function is a function $M$:[0, ∞)→[0, ∞) which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M can always be represented in the integral form $M(x) = \int_0^x q(t)dt$, where $q$, known as the Kernel of $M$, is right differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, $q(t) \to \infty$ as $t \to \infty$.

An Orlicz function is said to satisfy $\Delta_2$-condition for all values of $x$, if there exists a constant $K > 0$, such that $M(2x) \leq K\, M(x)$ for all $x \geq 0$. This condition is equivalent to $M(Lx) \leq KL\, M(x)$ for all values of $x \geq 0$ and for all $L \geq 1$ and $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

**Fuzzy Number: [18]:**

A fuzzy number X is a mapping $X: \mathbb{R} \longrightarrow [0,1]$ satisfying

(i) X is normal, i.e., ∃ an $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$;

(ii) X is convex, i.e., $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $r < t < s$;

(iii) X is upper semicontinuous i.e., for each $\epsilon > 0$, $X^{-1}([0, a+\epsilon])$ is open in the usual topology of $\mathbb{R}$, for all $a \in [0, 1]$;

(iv) $X^0 = \{t \in \mathbb{R}: X(t) > 0\}$ is compact.

In the present work, being motivated by the existing literature of statistical convergence of sequences of fuzzy numbers, the present author has defined the statistical convergence and lacunary statistical convergence on $F(L(\mathbb{R}), M, p, s)$ as follows:

**Statistical Convergence on $\mathbf{F(L(\mathbb{R}), M, p, s)}$:** A fuzzy number sequence $X = (X^{(n)}) = \left((X_k^{(n)})\right)$ in $F(L(\mathbb{R}), M, p, s)$ is said to converge statistically to $X_k = (X_1, X_2, \ldots\ldots)$, $X_k \in L(\mathbb{R})$, if for any $\varepsilon > 0$,

$$\delta\left(\left\{n \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n - X_k, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \geq \varepsilon\right\}\right) = 0$$

for each $k \in \mathbb{N}$. It is denoted by $x^{(n)} \xrightarrow{stat} L$.

A fuzzy number sequence $X = (X^{(n)}) = \left((X_k^{(n)})\right)$ in $F(L(\mathbb{R}), M, p, s)$ is said to be statistically Cauchy if for any $\varepsilon > 0$, ∃a natural number $m = m(\varepsilon)$ such that

$\delta\left(\left\{n \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n - X_k^m, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \geq \varepsilon\right\}\right) =$ 0for $n \geq m$ and for each $k \in \mathbb{N}$.

**Lacunary Convergence and Lacunary Statistical Convergence on $\mathbf{F(L(\mathbb{R}), M, p, s)}$**

The fuzzy number sequence $X = (X^{(n)}) = ((X_k^{(n)}))$inF(L(ℝ), M, p, s)is said to be$N_\theta$(L(ℝ), M, p, s)convergent to $X = (X_k)$ if for every $\varepsilon > 0$,

$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n - X_k, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \geq \varepsilon = 0$for n=1, 2,.....

The fuzzy number sequence $X = (X^{(n)}) = ((X_k^{(n)}))$in F(L(ℝ), M, p, s)is said to be $S_\theta$(L(ℝ), M, p, s)-convergent to $X = (X_k)$ if for every$\varepsilon > 0$,

$\lim_{r\to\infty} \frac{1}{h_r} C\left(\left\{n \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n - X_k, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \geq \varepsilon\right\}\right) = 0$for n-1, 2,.....

**Results**

**Theorem.** If $X = (X^{(n)}) = ((X_k^{(n)}))$ be a fuzzy number sequence in F(L(ℝ), M, p, s) ,then the following conditions are equivalent:

(a) $X^{(i)} \xrightarrow{stat} X$;

(b) There exists$Y^{(i)} = ((X_k^{(i)}))$and$\mathrm{Z}^{(i)} = ((\mathrm{Z}_k^{(i)}))$inF(L(ℝ), M, p, s)such that

$X^{(i)} = Y^{(i)} + \mathrm{Z}^{(i)} and\ \lim_{i\to\infty} Y^{(i)} = X$ in L(ℝ)and$\mathrm{Z}^{(i)} \xrightarrow{stat} \bar{0}$;

(c) There is a subsequence I = $\{i_r\}$ of ℕsuch that

$\delta(I) = 1$ and $\lim_{r\to\infty} X^{(i_r)} = X$ in L(ℝ)

where $X = (X_1, X_2, \dots.) \in$ L(ℝ).

Proof: (a)⇒ (b):

Let us select an increasing sequence of positive integers, where $N_0 = 0$, $N_0 < N_1 < N_2 < \cdots$. and if $i > N_j$,

$\delta\left(\left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(x_k^i - x_k, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \geq \frac{1}{j}\right\}\right) < 1/j$ for each $k \in \mathbb{N}$.

We define $Y^{(i)}$ and $\mathrm{Z}^{(i)}$ as follows:

If $N_0 < k < N_1$, $\mathrm{Z}_k^{(i)} = \bar{\theta}$ and $\mathrm{Y}_k^{(i)} = \mathrm{X}_k^{(i)}$ for each k.

If $N_m < i < N_{m+1}, m \geq 1$, set

$\mathrm{Y}_k^{(i)} = \mathrm{X}_k^{(i)}, \mathrm{Z}_k^{(i)} = \bar{\theta}$ if $\left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(x_k^n - x_k, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} < \frac{1}{m}$ and

$\mathrm{Y}_k^{(i)} = \mathrm{X}_i, \mathrm{Z}_k^{(i)} = \mathrm{X}_k^{(i)} - \mathrm{X}_i$ if $\left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(x_k^n - x_k, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \geq \frac{1}{m}$

Then $\mathrm{X}_k^{(i)} = \mathrm{Y}_k^{(i)} + \mathrm{Z}_k^{(i)}$ for each $i$ and $k$.

**Claim: $\lim_{i\to\infty} \mathbf{Y}_k^{(i)} = \mathbf{X}_k$.**

It follows from: For arbitrary $\varepsilon > 0$ we choose $m$ so that $\varepsilon > \frac{1}{\mathrm{m}}$.

Then for $i > N_m$, as $X^{(i)} \xrightarrow{stat} X$ holds, by construction

$$\left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n - X_k, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} < \frac{1}{m}$$

Since $\varepsilon$ is arbitrary, the claim is justified.

**Claim: $\mathbf{Z}^{(i)} \xrightarrow{stat} \bar{\mathbf{0}}$.**

Let $\eta > 0$ and m ∈ ℕ such that $\frac{1}{m} < \eta$.

It is trivially true that

$$\delta\left(\left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(\mathrm{Z}_k^i, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \neq 0\right\}\right) \geq \delta\left(\left\{i: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(\mathrm{Z}_k^i, \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \geq \varepsilon\right\}\right)$$

Let $\eta > 0$ and $m \in \mathbb{N}$ such that $\frac{1}{m} < \eta$.

Then from the construction it follows that, if$N_m < i \leq N_{m+1}$, then $Z_k^i \neq \bar{0}$ if and only if

$$\left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^i - X_k,\ \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} > \frac{1}{m}$$

So, if $N_p < i < N_{p+1},\ p \geq 1$,

$$\left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(Z_k^i,\ \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \neq 0\right\} \subseteq \left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^i - X_k,\ \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} > \frac{1}{p}\right\}$$

Consequently, if
$N_p < i < N_{p+1}, p > m$ ,since $M$ is non-decreasing, it follows that

$$\delta\left(\left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(Z_k^i,\ \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \neq 0\right\}\right)$$

$$\leq \delta\left(\left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^i - X_k,\ \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} > \frac{1}{p}\right\}\right) < \frac{1}{p} < \frac{1}{m} < \eta$$

And our claim is justified.

**Proof: (b)⇒ (c) :** Let (b) hold and let us define $I = (i_r)$to be a subsequence of $\mathbb{N}$ such that $i \in I$ if and only if for any $\varepsilon$> o,

$$\delta\left(\left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(Z_k^i,\ \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \geq \varepsilon\right\}\right) = 0$$

Then $\delta(I) = \delta\left(\mathbb{N} - \left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(Z_k^i,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} < \varepsilon\right\}\right) = 1 - 0 = 1.$

Since $X_k^{(i_r)} = Y_k^{(i_r)}$ and $\lim_{r\to\infty} Y_k^{(i_r)} = X_k$,by the given condition, (c) holds.

Proof: (c)⇒ (a) : Let (c) hold .

Then for arbitrary $\varepsilon > 0$, there exist a subsequence I = $\{i_r\}$ of $\mathbb{N}$such that $\delta(I) = 1$ and $m$=$m(\varepsilon) \in \mathbb{N}$such that

$\delta\left(\left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^{(i_r)} - X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} < \varepsilon\right\}\right)$ for $r \geq m$.

Then the result follows from the fact that

$$\delta\left(\left\{i \in \mathbb{N}: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^i - X_k,\ \bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \geq \varepsilon\right\}\right)$$

$< \delta(\mathbb{N} - \{i_r : r > m\})$

=1-$\delta(\{i_r : r > m\})$

=1-1

=0

**Theorem.** Let $\theta = (k_r)$be a lacunary sequence. Then for $((X_k^n)) \in N_\theta(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s})$ and $(X_k) \in \mathrm{L}(\mathbb{R})$,

(i) $X_k^n \to X_k$ in $\in N_\theta(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s}) \Rightarrow X_k^n \to X_k$ in $\in S_\theta(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s})$

(ii) $((X_k^n)) \in \ell_\infty(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s})$ and if $X_k^n \to X_k$ in $\in S_\theta(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s})$ then
$X_k^n \to X_k$ in $\in N_\theta(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s})$.

(iii)$\ell_\infty(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s}) \cap N_\theta(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s}) = \ell_\infty(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s}) \cap S_\theta(\mathrm{L}(\mathbb{R}), \mathrm{M}, \mathrm{p}, \mathrm{s})$

**Proof: (i)⇒ (ii):** The result holds from the following inequalities

$$\frac{1}{h_r}\sum_{k\in I_r}\left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\geq\varepsilon}}[\varepsilon]^{1/T}$$

$$\geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\geq\varepsilon}}\min[\varepsilon^h,\varepsilon^H]^{1/T}$$

$$\geq \frac{1}{h_r}\min[\varepsilon^h,\varepsilon^H]^{1/T}\left|\left\{k\in I_r: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\right\}\right|$$

for $0<h=\inf p_k\leq p_k\leq H=\sup p_k$.

**Proof: (ii)**$\Rightarrow$ **(iii):** The result holds from the following inequalities$\frac{1}{h_r}\sum_{k\in I_r}\left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}}$

$$=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\geq\varepsilon}}\left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}$$

$$+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}<\varepsilon}}\left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}$$

$$\leq\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\geq\varepsilon}}\max[K^h,K^H]^{1/T}+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}<\varepsilon}}[\varepsilon]^{1/T}$$

$$\leq\max[K^h,K^H]^{1/T}\frac{1}{h_r}\left|\left\{k\in I_r: \left[g_F\left(k^{-s}\left[M\left(\frac{\bar{d}(X_k^n-X_k,\bar{0})}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\geq\varepsilon\right\}\right|+[\max\ (\varepsilon^h,\varepsilon^H)]^{1/T}$$

for $0<h=\inf p_k\leq p_k\leq H=\sup p_k$.

**Proof: (iii)**$\Rightarrow$ **(i):** Follows from (i)$\Rightarrow$ (ii) *and* (ii) $\Rightarrow$ (iii)

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