DIRECT SYSTEM OF 3-RINGS

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Abstract: This paper presents definition of 3-ring, examples of 3-rings, definition of direct system of 3-rings, target for a direct system of 3-rings, direct limit for the direct system of 3-rings and theorems on direct system of 3-rings.

Keywords: 3-ring, direct limit, direct system and target.

Definition: A commutative ring (R, +, ., 1) such that $x^3 = x$, $3x = 0$ for all x in R is called a 3-ring.

Note: (1) $x + x = -x$ for all x in a 3-ring R.

(2) Hereafter R-stands for a 3-ring.

Theorem: Suppose (R, +, ., 1) is a 3-ring. Then $B(R) = \{e \in R \mid e^2 = e\}$. Then B(R) is a Boolean algebra with $\wedge, \vee, (-)'$ defined by $e \wedge f = ef$, $e \vee f = (e + f) - ef$, $e' = 1 - e$ for all e, f∈B(R).

Theorem: (i) The Boolean algebra B(R) of idempotents of a 3-ring B(R) is a Boolean ring with ⊕,⊙ defined by $e \oplus f = e + f + ef$, $e \odot f = ef$ for all $e, f \in B(R)$. (ii) For every $a \in R$, $e_a = 2a - a^2$, $f_a = a - a^2$. And clearly $e_a.f_a = 0$; Then clearly $B(R) = \{e_a, f_a \mid a \in R\}$.

Theorem: Every element a in R has a unique decomposition (normal form of a) given by $a = e_a - f_a$.

Note: Suppose $a, b \in R$ where R is a 3-ring. Then $a = e_a - f_a$ where $e_a = 2a - a^2$, $f_a = a - a^2$. Then

(i) $e_{a+b} = e_a + e_b + e_ae_b + f_af_b - e_af_b - e_bf_a$

(ii) $f_{a+b} = f_a + f_b + e_ae_b + f_af_b - e_af_b - e_bf_a$

(iii) $e_{ab} = e_ae_b + f_af_b$

(iv) $f_{ab} = e_af_b + e_bf_a$.

Theorem: The following are equivalent.

(a) The category of Boolean rings.

(b) The category of Boolean algebras.

(c) The category of A* - algebras.

(d) The category of 3-rings.

Theorem: Suppose (B, ∧, ∨, (−)′, 0, 1) is a Boolean ring. $R(B) = \{(e,f) \mid e, f \in B, ef = 0\}$.

Then (R(B), +, ., 0, 1) is a 3-ring where +, ., 0, 1 are defined by

(i) $(e_1, f_1) + (e_2, f_2) = (e_1 + e_2 + e_1e_2 + f_1f_2 - e_1f_2 - e_2f_1, f_1 + f_2 + e_1e_2 + f_1f_2 - e_1f_2 - e_2f_1)$

(ii) $(e_1, f_1)(e_2, f_2) = (e_1e_2 + f_1f_2, e_1f_2 + e_2f_1)$

(iii) $1 = (1,0)$, $0 = (0,1)$.

Example: **3** = {0, 1, 2}. Then (**3**, +, ., 1) is a 3-ring where

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Theorem: Suppose X is a non empty set. Then ($\mathbf{3}^X$, +, ., 0, 1) is a 3-ring where

(i) $(f+g)(x) = f(x) + g(x)$

(ii) $(f.g)(x) = f(x).g(x)$

(iii) $0(x) = 0$

(iv) $1(x) = 1$ for all $x \in X$, $f, g \in \mathbf{3}^X$.

Theorem: Suppose $\{R_i \mid i \in I\}$ is a family of 3-rings. Define $R = \prod_{i\in I} R_i = \{a \mid a : I \to \bigcup_{i\in I} R_i, a(i) \in R_i\}$.

Define +, ˙, −, 0, 1 on R as follows:

(i) $(a+b)(i) = a(i) + b(i)$

(ii) $(a.b)(i) = a(i) . b(i)$

(iii) $(-a)(i) = -(a(i))$

(iv) $0(i) = 0$

(v) $1(i) = 1$ ∀ i∀I and a,b∈R.

Then R is a 3-ring.

Definition: A directed set Λ is a set with a preorder ≤

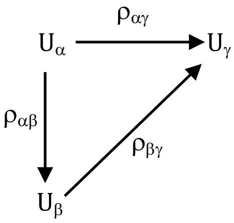
($\alpha \le \alpha$; $\alpha \le \beta$, $\beta \le \gamma \Rightarrow \alpha \le \gamma$) which also satisfies : $\forall\ \alpha, \beta \in \Lambda$ such that $\alpha \le \gamma$, $\beta \le \gamma$.

Note that $\Lambda_1 = \{(\alpha,\beta) \in \Lambda \times \Lambda \mid \alpha \le \beta\}$.

Definition: A direct system of sets indexed by a directed set Λ is a family $\{U_\alpha | \alpha \in \Lambda\}$ of sets together with, for each $(\alpha,\beta) \in \Lambda_1$,
a map of sets $\rho_{\alpha\beta} : U_\alpha \to U_\beta$ satisfying

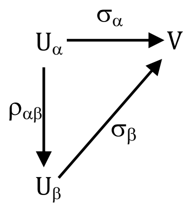
(a) $\forall\ \alpha \in \Lambda$, $\rho_{\alpha\alpha} = id_{U_\alpha}$

(b) $\forall\ \alpha, \beta, \gamma \in \Lambda$ with $\alpha \le \beta \le \gamma$ then



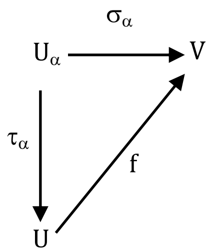
commutes, i.e., $\rho_{\alpha\gamma} = \rho_{\beta\gamma}\, \rho_{\alpha\beta}$.

Definition: Given a direct system, a target for the system is a set V and a collection of maps ($\sigma_\alpha : U_\alpha \to V$) satisfying the compatibility condition: $\forall\ \alpha \le \beta$



commutes, i.e., $\sigma_\alpha = \sigma_\beta\, \rho_{\alpha\beta}$.

Definition: A direct limit for the system is a target U, $(\tau_\alpha : U_\alpha \to U)_{\alpha\in\Lambda}$ satisfying the universal property: for any target V (with maps σ_α) there is a unique map $f: U \to V$ such that, $\forall\ \alpha \in \Lambda$



commutes, i.e., $\sigma_\alpha = f\, \tau_\alpha$

Theorem: Any two direct limits of a direct system are naturally isomorphic.

Notation: Direct limit is denoted by $\underset{\alpha\in\Lambda}{\overset{\text{limit}}{\longrightarrow}} U_\alpha$.

Theorem: Suppose U, $(\tau_\alpha : U_\alpha \to U)_{\alpha\in\Lambda}$ is a target for the system, $(U_\alpha)_{\alpha\in\Lambda}$, $(\rho_{\alpha\beta})_{(\alpha,\beta)\in\Lambda}$ such that

(i) $\forall\ u \in U$, $\exists\ \alpha \in \Lambda$ such that $u \in Im\tau_\alpha$

(ii) If $\alpha, \beta \in \Lambda$ and $u_\alpha \in U_\alpha$ and $u_\beta \in U_\beta$ then
$\tau_\alpha(u_\alpha) = \tau_\beta(u_\beta) \Leftrightarrow \exists\ \gamma \in \Lambda$ such that $\alpha \le \gamma$, $\beta \le \gamma$ and
$\rho_{\alpha\gamma}(u_\alpha) = \rho_{\beta\gamma}(u_\beta)$.

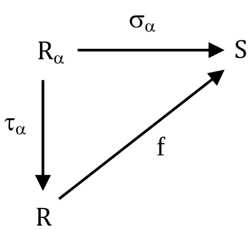
Then U is a direct limit of the system.

Theorem: Every direct system of sets has a direct limit.

Definition: A direct system of 3-rings is a direct system of sets $(R_\alpha)_{\alpha\in\Lambda}$, $(\rho_{\alpha\beta})_{(\alpha,\beta)\in\Lambda_1}$ such that each R_α has a 3-ring structure, and all $\rho_{\alpha\beta}$ are homomorphisms with these structures.

Definition: A target for a direct system of 3-rings is a target S, $(\sigma : R_\alpha \to S)_{\alpha\in\Lambda}$ for the underlying direct system of sets, together with a 3-ring structure on S such that all the σ_α are 3-ring homomorphisms.

Definition: A direct limit for the direct system of 3-rings is a target R, ($\tau_\alpha : R_\alpha \to R$) satisfying the universal property:
for any target S (with maps σ_α) there is a unique homomorphism $f : R \to S$ such that $\forall\ \alpha \in \Lambda$



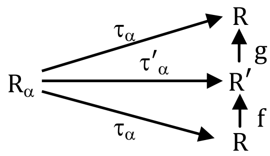
commutes, i.e., $f\, \tau_\alpha = \sigma_\alpha$.

Theorem: Given a direct system of 3-rings, any two direct limits for it are naturally isomorphic (as 3-rings).

Proof: Let $(R, (\tau_\alpha)_{\alpha\in\Lambda})$ and $(R', (\tau'_\alpha)_{\alpha\in\Lambda})$ be two direct limits of the direct systems of 3-rings.

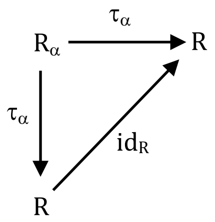
Since R is a direct limit and R′ is a target, we obtain an $f : R \to R'$; since R′ is direct limit, R is a target. We obtain $g : R' \to R$

$\forall\ \alpha \in \Lambda$



commutes..

But R is a target, and R is direct limit implies $id_R : R \to R$ is the unique map such that



commutes, hence $g \circ f = id_R$.

Similarly $f \circ g = id_{R'}$.

$\therefore R \cong R'$.

Theorem: Suppose $(R_\alpha)_{\alpha\in\Lambda}$, $(\rho_{\alpha\beta})_{(\alpha,\beta)\in\Lambda_1}$ is a direct system of 3-rings and R, $(\tau_\alpha)_{\alpha\in\Lambda}$ is a target satisfying

(i) $\forall\ r \in R\ \exists\ \alpha \in \Lambda$ such that $r \in Im\tau_\alpha$

(ii) $\forall\ \alpha \in \Lambda$, for $r_\alpha \in R_\alpha$ we have, $\tau_\alpha(r_\alpha) = 0 \Leftrightarrow \exists\ \beta \in \Lambda$ such that $\alpha \leq \beta$ and $\rho_{\alpha\beta}(r_\alpha) = 0$.

Then R is a direct limit of (R_α), $(\rho_{\alpha\beta})_{(\alpha,\beta)\in\Lambda_1}$.

Proof: Suppose S, $(\sigma_\alpha)_{\alpha\in\Lambda}$ is another target of (R_α), $(\rho_{\alpha\beta})_{(\alpha,\beta)\in\Lambda_1}$. Define $f : R \to S$ by :

let $r \in R \Rightarrow \exists\ \alpha \in \Lambda$ and $r_\alpha \in R_\alpha$ such that $\tau_\alpha(r_\alpha) = r$, then $f(r) = \sigma_\alpha(r_\alpha)$ …… (I)

<u>f is well defined</u>: since $0 \in R$, $\exists\ \alpha \in \Lambda$ such that $\tau_\alpha(r_\alpha) = 0$ (by (i))

$\because \tau_\alpha(r_\alpha) = 0 \Rightarrow f(0) = \sigma_\alpha(r_\alpha)$ (by (I))

By (ii) $\tau_\alpha(r_\alpha) = 0 \Rightarrow \exists\ \beta \in \Lambda$ such that $\alpha \leq \beta$ and $\rho_{\alpha\beta}(r_\alpha) = 0$.

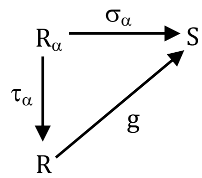
$\sigma_\alpha(r_\alpha) = \sigma_\beta\rho_{\alpha\beta}(r_\alpha) = \sigma_\beta(0) = 0$.

$\therefore \sigma_\alpha(r_\alpha) = 0$.

$\therefore f(0) = \sigma_\alpha(r_\alpha) = 0$.

$\therefore$ f is well defined.

<u>f is unique</u>: Suppose $g: R \to S$ is another homomorphism such that



commutes, i.e., $g\tau_\alpha = \sigma_\alpha$.

Let $r \in R \Rightarrow \exists\ \alpha \in \Lambda$ such that $\tau_\alpha(r_\alpha) = r$ (by (i))

$\Rightarrow f(r) = \sigma_\alpha(r_\alpha)$ (by (I))

$g(r) = g(\tau_\alpha(r_\alpha)) = \sigma_\alpha(r) = f(r)$.

$\therefore$ f is unique.

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