FEKETE-SZEGÖ INEQUALITY FOR A NEW CLASS OF ANALYTIC FUNCTIONS AND ITS SUBCLASS

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Abstract: We introduce a new class of analytic functions and its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes.

Keywords: *bounded functions, Inverse Starlike functions, Starlike functions and Univalent functions.*

Introduction: Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([8], [9]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [7] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and $a_2{}^2$ for the class $\mathcal{S}$, Fekete and Szegö [10] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, \text{ if } \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), \text{ if } 0 \leq \mu \leq 1; \\ 4\mu - 3, \text{ if } \mu \geq 1. \end{cases} \quad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (Chhichra[1], Babalola[7]).

Let us define some subclasses of $\mathcal{S}$.

We denote by $\mathcal{S}^*$, the class of univalent starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A}$ satisfying the condition

$\boldsymbol{Re}\frac{((zh'(z))}{h'(z)} > 0, \boldsymbol{z} \in \mathbb{E}.$ (1.4)

We introduce the class $\mathcal{A}(\alpha,\beta)$ of functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$\boldsymbol{Re}\left[(1-\alpha-\beta)f'(z) + \alpha\frac{zf'(z)}{f(z)} + \beta\frac{\{zf'(z)\}'}{f'(z)}\right] > 0, \boldsymbol{z} \in \mathbb{E}$$

i.e. $(1-\alpha-\beta)f'(z) + \alpha\frac{zf'(z)}{f(z)} + \beta\frac{\{zf'(z)\}'}{f'(z)} \prec \frac{1+z}{1-z}$ (1.5)

The subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$(1-\alpha-\beta)f'(z) + \alpha\frac{zf'(z)}{f(z)} + \beta\frac{\{zf'(z)\}'}{f'(z)} \prec \frac{1+Az}{1+Bz}; -1 \le B \le A \le 1$ (1.6)

is denoted by $\mathcal{A}(\alpha,\beta;A,B)$.

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is denoted by $\mathcal{A}(\alpha,\beta;\delta)$.

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is denoted by $\mathcal{A}(\alpha,\beta;A,B;\delta)$.

It is to be noted that

- $\mathcal{A}(\alpha,\beta;A,B;1) = \mathcal{A}(\alpha,\beta;A,B)$
- $\mathcal{A}(\alpha,\beta;1,-1;\delta) = \mathcal{A}(\alpha,\beta;\delta)$
- $\mathcal{A}(\alpha,\beta;1,-1) = \mathcal{A}(\alpha,\beta)$
- $\mathcal{A}(\alpha,\beta;1) = \mathcal{A}(\alpha,\beta)$
- $\mathcal{A}(1,0) = S^*$
- $\mathcal{A}(0,1) = \mathcal{K}$

Symbol $\prec$ stands for subordination, which we define as follows:

Principle Of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to $F(z)$ in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.9)

It is known that $|d_1| \le 1, |d_2| \le 1 - |d_1|^2$.

Main Results:

Theorem 2.1: If $f(z)\in\mathcal{A}(\alpha,\beta)$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\left(\frac{2}{2-\alpha}\right)^2\left\{\frac{(4-2\alpha+\alpha^2+8\beta)}{2(3-\alpha+3\beta)}-\mu\right\}, if\ \mu\le\frac{(\alpha+4\beta)}{(3-\alpha+3\beta)}; & (2.1)\\ \frac{2}{(3-\alpha+3\beta)}, if\ \frac{(\alpha+4\beta)}{(3-\alpha+3\beta)}\le\mu\le\frac{4-3\alpha+\alpha^2+4\beta}{(3-\alpha+3\beta)}; & (2.2)\\ \left(\frac{2}{2-\alpha}\right)^2\left\{\mu-\frac{(4-2\alpha+\alpha^2+8\beta)}{2(3-\alpha+3\beta)}\right\}, if\ \mu\ge\frac{4-3\alpha+\alpha^2+4\beta}{(3-\alpha+3\beta)}. & (2.3)\end{cases}$$

The results are sharp.

Proof: By definition of $\mathcal{A}(\alpha,\beta)$, we have

$$(1-\alpha-\beta)f'(z)+\alpha\frac{zf'(z)}{f(z)}+\beta\frac{\{zf'(z)\}'}{f'(z)}\prec\frac{1+z}{1-z} \quad (2.4)$$

Expanding (2.4), we have

$$1+(2-\alpha)a_2z+\left\{(3-\alpha+3\beta)a_3-(\alpha+4\beta){a_2}^2\right\}z^2+---= (1+2c_1z+2(c_2+c_1^2)z^2+---) \quad (2.5)$$

Identifying terms in (2.5), we get

$$\begin{cases}a_2=\frac{2}{2-\alpha}c_1\\ a_3=\frac{2}{(3-\alpha+3\beta)}\left[c_2+\left\{\frac{4-2\alpha+\alpha^2+8\beta}{(2-\alpha)^2}\right\}c_1^2\right]\end{cases} \quad (2.6)$$

Using (2.6), we obtain

$$a_3-\mu a_2^2=\frac{2}{(3-\alpha+3\beta)}c_2+\left\{\frac{2(4-2\alpha+\alpha^2+8\beta)}{(3-\alpha+3\beta)(2-\alpha)^2}-\frac{4\mu}{(2-\alpha)^2}\right\}c_1^2$$

This leads to

$$|a_3-\mu a_2^2|\le\frac{2}{(3-\alpha+3\beta)}+\left\{\left|\frac{2(4-2\alpha+\alpha^2+8\beta)}{(3-\alpha+3\beta)(2-\alpha)^2}-\frac{4\mu}{(2-\alpha)^2}\right|-\frac{2}{(3-\alpha+3\beta)}\right\}|c_1|^2 \quad (2.7)$$

Case I: $\mu\le\frac{(4-2\alpha+\alpha^2+8\beta)}{2(3-\alpha+3\beta)}$, we get from (2.7)

$$\begin{aligned}|a_3-\mu a_2^2|&\le\frac{2}{(3-\alpha+3\beta)}+\left\{\frac{2(4-2\alpha+\alpha^2+8\beta)}{(3-\alpha+3\beta)(2-\alpha)^2}-\frac{4\mu}{(2-\alpha)^2}-\frac{2}{(3-\alpha+3\beta)}\right\}|c_1|^2\\ &=\frac{2}{(3-\alpha+3\beta)}+\left\{\frac{4(\alpha+4\beta)}{(3-\alpha+3\beta)(2-\alpha)^2}-\frac{4\mu}{(2-\alpha)^2}\right\}|c_1|^2\\ &=\frac{2}{(3-\alpha+3\beta)}+\left(\frac{2}{2-\alpha}\right)^2\left\{\frac{(\alpha+4\beta)}{(3-\alpha+3\beta)}-\mu\right\}|c_1|^2 \quad (2.8)\end{aligned}$$

Subcase I(a): $\mu\le\frac{(\alpha+4\beta)}{(3-\alpha+3\beta)}$. From equation (2.8), we get

$$\begin{aligned}|a_3-\mu a_2^2|&\le\frac{2}{(3-\alpha+3\beta)}+\left(\frac{2}{2-\alpha}\right)^2\left\{\frac{(\alpha+4\beta)}{(3-\alpha+3\beta)}-\mu\right\}\\ &=\left(\frac{2}{2-\alpha}\right)^2\left\{\frac{(4-2\alpha+\alpha^2+8\beta)}{2(3-\alpha+3\beta)}-\mu\right\} \quad (2.9)\end{aligned}$$

Subcase I(b): $\mu \geq \frac{(\alpha+4\beta)}{(3-\alpha+3\beta)}$.

From equation (2.8), we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{(3-\alpha+3\beta)} \qquad (2.10)$$

Case II: $\mu \geq \frac{(4-2\alpha+\alpha^2+8\beta)}{2(3-\alpha+3\beta)}$, we get from (2.7)

$$|a_3 - \mu a_2^2| \leq \frac{2}{(3-\alpha+3\beta)} + \left\{\frac{4\mu}{(2-\alpha)^2} - \frac{2(4-3\alpha+\alpha^2+4\beta)}{(3-\alpha+3\beta)(2-\alpha)^2}\right\}|c_1|^2 \qquad (2.11)$$

Subcase II(a): $\mu \leq \frac{4-3\alpha+\alpha^2+4\beta}{(3-\alpha+3\beta)}$. From equation (2.11), we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{(3-\alpha+3\beta)} \qquad (2.12)$$

Combining subcase I(a) and subcase II(b), we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{(3-\alpha+3\beta)}, if \ \frac{(\alpha+4\beta)}{(3-\alpha+3\beta)} \leq \mu \leq \frac{4-3\alpha+\alpha^2+4\beta}{(3-\alpha+3\beta)}. \qquad (2.13)$$

Subcase II(b): $\mu \geq \frac{4-3\alpha+\alpha^2+4\beta}{(3-\alpha+3\beta)}$. From equation (2.11), we get

$$|a_3 - \mu a_2^2| \leq \left(\frac{2}{2-\alpha}\right)^2 \left\{\mu - \frac{(4-2\alpha+\alpha^2+8\beta)}{2(3-\alpha+3\beta)}\right\}. \qquad (2.14)$$

This completes the theorem. The results are sharp.

Theorem 2.2: If $f(z) \in \mathcal{A}(\alpha,\beta;A,B)$, then

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{1}{(2-\alpha)^2}[\{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta\} - 2(A-B)(3-\alpha+3\beta)\mu], \\ \qquad if \ \mu \leq \frac{2(A+B)\alpha - B(4-\alpha^2)+8(A-B)\beta-(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)} \\ \frac{2(A-B)}{|3-\alpha+3\beta|}, if \ \mu \leq \frac{2(A+B)\alpha - B(4-\alpha^2)+8(A-B)\beta-(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)}. \leq \mu \leq \frac{2(A+B)\alpha - B(4-\alpha^2)+8(A-B)\beta+(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)}; \\ \frac{1}{(2-\alpha)^2}[-2(A-B)(3-\alpha+3\beta)\mu - \{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta\}],. \\ \qquad if \ \mu \geq \frac{2(A+B)\alpha - B(4-\alpha^2)+8(A-B)\beta+(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)} \end{cases}$$

The results are sharp.

Proof: By definition of $\mathcal{A}(\alpha,\beta;A,B)$, we have

$$(1-\alpha-\beta)f'(z) + \alpha\frac{zf'(z)}{f(z)} + \beta\frac{\{zf'(z)\}'}{f'(z)} \prec \frac{1+Az}{1+Bz}; -1 \leq B \leq A \leq 1 \qquad (2.18)$$

Expanding (2.18), we have

$$1 + (2-\alpha)a_2 z + \{(3-\alpha+3\beta)a_3 - (\alpha+4\beta)a_2^2\}z^2 + --- = (1 + 2(A-B)c_1 z + 2(A-B)(c_2 - Bc_1^2)z^2 + ---) \qquad (2.19)$$

Identifying terms in (2.19), we get

$$\begin{cases} a_2 = \frac{2(A-B)}{2-\alpha} c_1 \\ a_3 = \frac{2(A-B)}{(3-\alpha+3\beta)}\left[c_2 + \left\{\frac{2(A+B)\alpha-4B-B\alpha^2+8(A-B)\beta}{(2-\alpha)^2}\right\} c_1^2\right] \end{cases} \quad (2.20)$$

Using (2.20), we obtain

$$\frac{(3-\alpha+3\beta)}{2(A-B)}\left(a_3-\mu a_2^2\right) = c_2 + \frac{1}{(2-\alpha)^2}\left\{\left\{2(A+B)\alpha - B\left(4-\alpha^2\right) + 8(A-B)\beta\right\} - 2(A-B)(3-\alpha+3\beta)\mu\right\} c_1^2$$

This leads to

$$\frac{|3-\alpha+3\beta|}{2(A-B)}\left|a_3-\mu a_2^2\right| \le 1 + \frac{1}{(2-\alpha)^2}\left\{\left|\left\{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta\right\} - 2(A-B)(3-\alpha+3\beta)\mu\right| - (2-\alpha)^2\right\}|c_1|^2 \quad (2.21)$$

Case I: $\mu \le \frac{2(A+B)\alpha - B(4-\alpha^2)+8(A-B)\beta}{2(A-B)(3-\alpha+3\beta)}$, we get from (2.21)

$$\frac{|3-\alpha+3\beta|}{2(A-B)}\left|a_3-\mu a_2^2\right| \le 1 + \frac{1}{(2-\alpha)^2}\left\{\left\{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta - (2-\alpha)^2\right\} - 2(A-B)(3-\alpha+3\beta)\mu\right\}|c_1|^2 \quad (2.22)$$

Subcase I(a): $\mu \le \frac{2(A+B)\alpha - B(4-\alpha^2)+8(A-B)\beta-(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)}$. From equation (2.22), we get

$$\frac{|3-\alpha+3\beta|}{2(A-B)}\left|a_3-\mu a_2^2\right| \le 1 + \frac{1}{(2-\alpha)^2}\left\{\left\{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta - (2-\alpha)^2\right\} - 2(A-B)(3-\alpha+3\beta)\mu\right\} = \frac{1}{(2-\alpha)^2}\left[\left\{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta\right\} - 2(A-B)(3-\alpha+3\beta)\mu\right]$$

$$\left|a_3-\mu a_2^2\right| \le \frac{2(A-B)}{|3-\alpha+3\beta|(2-\alpha)^2}\left[\left\{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta\right\} - 2(A-B)(3-\alpha+3\beta)\mu\right] \quad (2.23)$$

Subcase I(b): $\mu \ge \frac{2(A+B)\alpha - B(4-\alpha^2)+8(A-B)\beta-(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)}$.

From equation (2.22), we get $\left|a_3-\mu a_2^2\right| \le \frac{2(A-B)}{|3-\alpha+3\beta|}$ (2.24)

Case II: $\mu \ge \frac{2(A+B)\alpha - B(4-\alpha^2)+8(A-B)\beta-(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)}$, we get from (2.21)

$$\frac{|3-\alpha+3\beta|}{2(A-B)}\left|a_3-\mu a_2^2\right| \le 1 + \frac{1}{(2-\alpha)^2}\left[2(A-B)(3-\alpha+3\beta)\mu - \left\{2(A+B)\alpha - B\left(4-\alpha^2\right) + 8(A-B)\beta + (2-\alpha)^2\right\}\right]|c_1|^2 \quad (2.25)$$

Subcase II(a): $\mu \le \frac{2(A+B)\alpha - B\left(4-\alpha^2\right)+8(A-B)\beta+(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)}$. From equation (2.25), we get

$\left|a_3 - \mu a_2^2\right| \le \frac{2(A-B)}{|3-\alpha+3\beta|}$ (2.26)

Combining subcase I(a) and subcase II(b), we get

$\frac{|3-\alpha+3\beta|}{2(A-B)}\left|a_3 - \mu a_2^2\right| \le 1, if\ \frac{2(A+B)\alpha-B(4-\alpha^2)+8(A-B)\beta-(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)} \le \mu \le \frac{2(A+B)\alpha-B(4-\alpha^2)+8(A-B)\beta+(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)}.$ (2.27)

Subcase II(b): $\mu \ge \frac{2(A+B)\alpha-B(4-\alpha^2)+8(A-B)\beta+(2-\alpha)^2}{2(A-B)(3-\alpha+3\beta)}$. From equation (2.25), we get

$\frac{|3-\alpha+3\beta|}{2(A-B)}\left|a_3 - \mu a_2^2\right| \le \frac{1}{(2-\alpha)^2}\left[2(A-B)(3-\alpha+3\beta)\mu - \{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta\}\right]$

$\left|a_3 - \mu a_2^2\right| \le \frac{2(A-B)}{|3-\alpha+3\beta|(2-\alpha)^2}\left[2(A-B)(3-\alpha+3\beta)\mu - \{2(A+B)\alpha - B(4-\alpha^2) + 8(A-B)\beta\}\right]$ (2.28)

This completes the theorem. The results are sharp.

Corollary2.4: Putting $\mathbf{A = 1, B = -1}$ in theorem 2.3, we get

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \left(\frac{2}{2-\alpha}\right)^2\left\{\frac{(4-2\alpha+\alpha^2+8\beta)}{2(3-\alpha+3\beta)} - \mu\right\}, if\ \mu \le \frac{(\alpha+4\beta)}{(3-\alpha+3\beta)}; \\ \frac{2}{(3-\alpha+3\beta)}, if\ \frac{(\alpha+4\beta)}{(3-\alpha+3\beta)} \le \mu \le \frac{4-3\alpha+\alpha^2+4\beta}{(3-\alpha+3\beta)}; \\ \left(\frac{2}{2-\alpha}\right)^2\left\{\mu - \frac{(4-2\alpha+\alpha^2+8\beta)}{2(3-\alpha+3\beta)}\right\}, if\ \mu \ge \frac{4-3\alpha+\alpha^2+4\beta}{(3-\alpha+3\beta)}. \end{cases}$$

, which is the required result for the class $f(z) \in \mathcal{A}(\alpha, \beta)$.

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