A NUMERICAL SOLUTION OF THE POINT KINETICS EQUATIONS AND ITS VALIDATION

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Abstract: The present paper uses Generalized Runge-Kutta (GRK) method for numerical solution of point kinetic equations to estimate transient power variation of supercritical water reactor. The system of the point kinetic equations possesses a stiff character and nonlinear in nature. A numerical model is developed to solve point kinetic equations and then validated against available analytical and numerical results. The results obtained with the present model are compared for different case studies associated with varieties of reactivity insertion or removal incidents. The comparisons show a good agreement among the results.

Keywords: Nuclear coupled thermal hydraulics, Point kinetics, Reactivity, Supercritical water-cooled reactor

Introduction: In nuclear water reactor, power generation occurs via fission of nuclear fuel. The fission rate and subsequently heat generated in the reactor core is directly related to the neutron population which in turn depends on material of nuclear fuel and the thermodynamic properties of the coolant, moderator and fuel. The point reactor kinetic equations with non-linear relationship multi-group delayed neutrons precursors are useful in providing power estimates and other system variables of reactor core. The solution of these equations provide insight into transient behaviour of reactor core, for example, in understanding the power fluctuations experienced during start-up or shut-down when the control rods are adjusted. Recently, a large number of attempts (Aboanber [4], Ambrosini et al. [5], and Nahla [6]) have been made to model the time dependent behaviour of a nuclear reactor using point kinetic equations.

In the present paper the Runge-Kutta method of order four is applied for solving point kinetics equations. The organisation of this paper is as follows. After the general introduction in section 1, the detail of the mathematical formulation is provided in section 2. In section 3, the model

developed is applied to the coupled nonlinear stiff system of point kinetic equations. In section 4, various case studies are undertaken and the transient simulations are carried out with the present model. The results obtained are validated with available benchmarks.

Nomenclature

English letters

N(t)	Relative power
C_i	Concentration of i^{th} delayed neutron precursor group
t	Time
h	Step size
T	Temperature
T_o	Initial Temperature

Greek symbols

β_i	Delayed neutron fraction of i^{th} group
β	Total delayed neutron fraction = $\sum_{i=1}^{6} \beta_i$ i
Λ	Prompt neutron generation time
λ_i	Decay constant of i^{th} group
ρ	Reactivity
ρ_o	Initial reactivity
α	Temperature coefficient of reactivity

Subscripts

i^{th} delayed neutron precursor group

Mathematical Formulation: In the point kinetic set of seven coupled ordinary differential equations describe the change in relative power and decay of six delayed neutrons precursor group, which are written as:

$$\frac{\partial N(t)}{\partial t} = \left(\frac{\rho - \beta}{\Lambda} \right) N(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \qquad (1)$$

$$\frac{\partial C_i(t)}{\partial t} = \frac{\beta_i}{\Lambda} N(t) - \lambda_i C_i(t) \qquad (2)$$

These equations can be written in other form as follows:

$$\frac{\partial}{\partial t}\left[\underline{y}\right] = \underline{D}\left[\underline{y}\right] \qquad (3)$$

Where $\underline{y}$ is a vector of unknown variables [N(t) C_1(t) C_2(t) C_3(t) C_4(t) C_5(t) C_6(t)]^T, $\underline{D}$ is square matrix of coefficients.

Generalized Runge–Kutta method (GRK)

A generalized Runge–Kutta (GRK) method for the numerical solution of the initial value problem

$$\underline{y}'(x) = \underline{f}[\underline{y}(x)], \quad \underline{y}\ (x_0) = \underline{y}_0 \qquad (4)$$

$$\underline{y}_{i+1} = \underline{y}_i + \sum_{j=1}^{s} \hat{C}_j K_j \qquad (5)$$

where $\underline{y}$, $\underline{y}_0$ and $\underline{f}$ are vectors in n-dimensional real space.

Determination of constants

Kaps and Rentrop [4] used Rosenbrock-Wanner method for determination of coefficients, which is as follows:

$$\left[\underline{\underline{I}} - \gamma h \underline{\underline{f}}'(\underline{y}_i)\right] K_i = h\underline{f}\left(\underline{y}_i + \sum_{j=1}^{i-1} \alpha_{ij} K_j\right) + h\underline{\underline{f}}'(\underline{y}_i) \sum_{j=1}^{i-1} \gamma_{ij} K_j \qquad i = 1,\ldots\ldots,s \qquad (6)$$

This method coincides with the corresponding Runge–Kutta method when γ_{ij} equals to zero. The vector K_j is determined by solving s number of linear equations. Here s is equal to the order of the method. To evaluate the K_j from (6), the set of equations are:

$$\left.\begin{aligned} K_1 &= h\underline{\underline{B}}\,\underline{f}(\underline{y}) \\ K_2 + \frac{\gamma_{21}}{\gamma}K1 &= \underline{\underline{B}}\left[h\underline{f}(\underline{y}+\alpha_{21}K1) + \frac{\gamma_{21}}{\gamma}K_1 \right] \\ K_3 + \frac{1}{\gamma}\sum_{l=1}^{2}\gamma_{3l}K_l &= \underline{\underline{B}}\left[h\underline{f}\left(\underline{y}+\sum_{l=1}^{2}\alpha_{3l}K_l \right) + \frac{1}{\gamma}\sum_{l=1}^{2}\gamma_{3l}K_l \right] \\ K_4 + \frac{1}{\gamma}\sum_{l=1}^{3}\gamma_{4l}K_l &= \underline{\underline{B}}\left[h\underline{f}\left(\underline{y}+\sum_{l=1}^{3}\alpha_{4l}K_l \right) + \frac{1}{\gamma}\sum_{l=1}^{3}\gamma_{4l}K_l \right] \end{aligned}\right\} \quad (7)$$

$$\underline{\underline{B}} = [\underline{\underline{I}} - h\gamma \underline{\underline{f}}'(\underline{y})]^{-1}$$

The quantities C_j, γ, α_{ij}, and γ_{ij} are real numbers. In this case same values as used in [1] are considered for present simulation, which are as follows

Table I: List of coefficients

γ=0.395				
	γ_{ij}	α_{ij}		$\hat{C}_1 = 0.199\ 293\ 275\ 701$
ij=21	-0.767 672 395 484	-		$\hat{C}_2 = 0.482\ 645\ 235\ 674$
ij=31	-0.851 675 323 742	0.796 920457 938		$\hat{C}_3 = 0.068\ 061\ 488\ 625\ 6$
ij=32	0.522 967 289 188	0.073 079 542 061 5		$\hat{C}_4 = 0.25$
ij=41	0.288 463 109 545	0.796 920457 938		$\overline{C_1} = 0.346\ 325\ 833\ 758$
ij=42	0.088 021 427 338 1	0.073 079 542 061 5		$\overline{C_1} = 0.285\ 693\ 175\ 712$
ij=43	-0.337 389 840 627	0		$\overline{C_1} = 0.367\ 980\ 990\ 53$

Step size control

Truncation error is calculated by combining third order method with forth order method. If forth and third order approximations are denoted by $\underline{y}_{i+1}$ and $\overline{\underline{y}}_{i+1}$, respectively. Then according to (4)

$$\underline{y}_{i+1} = \underline{y}_i + \sum_{j=1}^{s}\hat{C}_j K_j \quad \text{for (s=4)} \quad (8)$$

and $\underline{\bar{y}}_{i+1} = \underline{\bar{y}}_i + \sum_{j=1}^{s} \bar{C}_j K_j$ for (s=3) (9)

are combined, where constants have values given in table 1. The strategy proposed by Kaps and Rentrop [4] for step size control is as following:

$$\varepsilon = \max_i \frac{\left|\bar{y}^i(xold) - y^i(xold)\right|}{S_i} \qquad (10)$$

$$S_i = \max_i \left\{1, \left|\bar{y}^i(x_j)\right|\right\} \qquad (11)$$

and if $\varepsilon < \delta$ then,

$$\bar{h}_{new} = 0.09 h_{old} (\delta / \varepsilon)^{1/4} \qquad (12)$$

$$h_{new} = 0.5 h_{old} \quad \text{if} \quad \bar{h}_{new} \le 0.5 h_{old}$$

$$h_{new} = \bar{h}_{new}, \quad \text{if} \quad 0.5 < \bar{h}_{new} < 1.5 h_{old}$$

$$h_{new} = 1.5 h_{old} \quad \text{if} \quad \bar{h}_{new} \ge 1.5 h_{old}$$

otherwise, $\bar{h}_{new}$ is calculated again. This process is repeated until $h_{new} \le h_{\min}$, a chosen minimal step size which depends upon machine precision and length of integration interval.

Solution of Point Kinetic Equations: A dependent variable τ is added to the system in order to make the system dimension $n \times n$ matrix.

$$\frac{d\tau}{dt} = 1 \qquad (13)$$

With (1), (2) and (13)

$$\underline{f}(\underline{y}) = \begin{bmatrix} \dfrac{\rho(y^8) - \beta}{\Lambda} y^1 + \sum_{j=1}^{6} \lambda_j y^{j+1} \\ \dfrac{\beta_1}{\Lambda} y^1 - \lambda_1 y^2 \\ \vdots \\ \dfrac{\beta_6}{\Lambda} y^1 - \lambda_6 y^6 \\ 1 \end{bmatrix} \qquad (14)$$

and then Jacobian matrix

$$\underline{\underline{f}}'(\underline{y}) = \begin{bmatrix} \frac{1}{\Lambda}[\rho(y^8)-\beta] & \lambda_1 & \lambda_2 & \dots & \lambda_6 & \frac{y^1}{\Lambda}\frac{\partial\rho}{\partial y^8} \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & \dots & 0 & 0 \\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \\ \frac{\beta_6}{\Lambda} & 0 & 0 & \dots & -\lambda_6 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad (15)$$

If $\underline{\underline{A}} = [\underline{\underline{I}} - p\underline{\underline{f}}'(\underline{y})]$ where $p = \gamma/h$, we get

$$\underline{\underline{A}} = \begin{bmatrix} 1-[\rho(y^8)-\beta]\frac{p}{\Lambda} & -p\lambda_1 & -p\lambda_2 & \dots & p\lambda_6 & \frac{p}{\Lambda}y^1\frac{\partial\rho}{\partial y^8} \\ \frac{\beta_1}{\Lambda} & 1+p\lambda_1 & 0 & \dots & 0 & 0 \\ \frac{\beta_2}{\Lambda} & 0 & 1+p\lambda_2 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \\ \frac{\beta_6}{\Lambda} & 0 & 0 & & 1+p\lambda_6 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad (16)$$

Determination of $\underline{\underline{B}}$ can be done by taking inverse of $\underline{\underline{A}}$. Once $\underline{\underline{B}}$ is calculated K_j can be calculated using (6). Now we can calculate $\underline{y}_{i+1}$ from (4).

Validation of the Numerical Model: The present section deals with different case studies, where point kinetic equations for six-group delayed neutron precursor groups are solved to simulate various reactivity insertion or removal incidents. Reactivity in the case studies undertaken,

varies step-wise, ramp-wise , zig-zag manner or depending upon temperature feedback effects, with N (o) =1 and i-group of delayed neutron precursor concentration

C_i = (N (o) β_i)/($\Lambda \lambda_i$) define the initial condition for the transient simulations. The input parameters λ_i and β_i considered in the various case studies are listed in table II.

Table II: Parameters of the point kinetics

Precursor group (i)	i=1	i=2	i=3	i=4	i=5	i=6
For step reactivity case I (Sanchez[1]) & ramp reactivity, β=0.007, Λ =0.0002s						
Decay constant (λ_i)	0.0127	0.0317	0.115	0.311	1.4	3.87
Delayed neutron fraction (β_i)	0.000266	0.001491	0.001316	0.002849	0.000896	0.000182
For step reactivity case II & zigzag reactivity, β=0.0075, Λ =0.0005s						
Decay constant (λ_i)	0.0127	0.0317	0.115	0.311	1.4	3.87
Delayed neutron fraction (β_i)	0.000285	0.001597 5	0.00141	0.003052 5	0.00096	0.000165
For temperature feedback reactivity, β=0.00645, Λ =0.00005s						
Decay constant (λ_i)	0.0124	0.030	0.111	0.301	1.13	3.0
Delayed neutron fraction (β_i)	0.00021	0.00141	0.00127	0.00255	0.00074	0.00645

Step reactivity

To check the accuracy of the present model, two case studies associated with step change in reactivity have been considered with different values of data set.

Case I. Relative power, N(t) is obtained for different step reactivity with the present model. The results are shown in Table III and compared with the same as obtained by Sanchez[1].

Case II . For the different values of step reactivity -0.10$, -0.5$, +0.5$ and +1.0$, the relative power, N(t) is obtained with the present model and compared with Nahla [6]. The comparisons in relative power are presented in Table III.

Ramp reactivity : In the present case study, the parameters listed in Table II are used to produce results for ramp reactivity. The relative power obtained with the model are compared with those values obtained by Nahla [6] as well as for Sanchez [1]. Two cases of the reactivity, a positive ramp with $ 0.1/s and a negative ramp with $ 0.1/s, are introduced in Table IV.

Zig-zag reactivity: A calculation has been performed with six delayed groups involved a zig-zag ramp with slope as given in Table V.

Table III: Performance of present model for step reactivity

Case I				Case II			
$\rho_{reactivit}$	Time (s)	RP,N(t) Present Model	RP,N(t) Sanchez[1]	$\rho_{reactivity}$	Time, t (s)	N(t) Present Model	N(t) Nahla [6]
				-1.0$	0.1	0.63621	0.521
0.003	1	2.2098	2.20985		1	0.43572	0.433
	10	8.0192	8.01891		10	0.23661	0.2.36
0.0055	0.1	5.2100	5.21	-0.5$	0.1	0.7913	0.6.99
	2	4.3025E01	4.3022E01		1	0.60942	0.607
	10	1.3886E05	1.388E05		10	0.39673	0.396
0.007	0.01	4.5089	4.50885	0.5$	0.1	1.2834	1.53
	0.5	5.3459E03	5.4335E03		1	2.477	2.510
	2	2.0592E11	2.05697E11		10	14.0841	14.2
0.008	0.01	6.2029	6.20276	1.0$	0.1	1.6714	2.520
	0.1	1.4104E03	1.14101E03		0.5	8.99613	10.401
	1	6.6133E23	6.1486E23		10	2.90E-07	3.22E+01

Table IV: Performance of present model for ramp reactivity

Time(s	2	4	6	8	9	10	11
RP, N(t) for $0.1/s ramp reactivity							
Present Model	1.3381	2.2282	5.5818	4.2778E 01	4.8739E 02	4.5095E 05	1.7906E1 6
Nahla [5]	1.3382	2.2284	5.582	4.2786E 01	4.8751E 02	4.5115E 05	1.7904E1 6
Sanche z [1]	1.3382	2.2284	5.582	4.2786E 01	4.8751E 02	4.5115E 05	1.7904E1 6
RP, N(t) for -$0.1/s ramp reactivity							
Present Model	0.0792 02	0.6132 0	0.4740 6	0.36917	-	0.29066	-
Nahla [5]	0.7919 55	0.6129 76	0.4740 27	0.36914 5	-	0.29063 6	-

Table V: Performance of present model for zig-zag ramp

Zig-zag ramp slope					
Time range	$0 \le t < 0.5$	$0.5 \le t < 1$	$1 \le t < 1.5$	$t \ge 1.5$	-
Slope	$1/s	-$1/s	$1/s	0	-
Time, t (s)	0.5	1	1.5	2	10
N(t), Present Model	1.7211	1.2113	1.8919	2.5215	1.2047E01
N(t), Sanchez [1]	1.7213	1.2110	1.8921	2.5216	1.2046E01

Temperature feedback reactivity

In this case, the present model is applied to solve the point reactor kinetic equations in the presence of temperature feedback reactivity. The equations for temperature and reactivity relationship are as given below:

$$\rho(t) = \rho(t-1) - \alpha[T(t) - T(t-1)] \qquad (17)$$

$$\frac{dT(t)}{dt} = K_C N(t) \qquad (18)$$

The relative power obtained with the model are compared with those

obtained by Nahla [6] for initial value of reactivity taken as 0.5$, 1.0$, 1.2$, 1.5$ and 2$. The values of $\alpha = 5\times10^{-5}\ K^{-1}$, and the reciprocal of the thermal capacity of reactor K_c = 0.05 K/MWs are taken for this case. The peak value of the relative power and the time of its occurring for both the models are listed in Table VI. The temporal variation of relative power and reactivity with both the models at initial reactivity 0.2 $ are shown in Fig 1

Table VI: Performance of present model for temperature feedback reactivity

Initial reactivity, ρ_0 ($)		0.5	1	1.2	1.5	2
Present Model	Peak RP, N(t)	45.7546	809.2118	8094.549	43975.023	175213.14
	Peak time, t (s)	28.2940	0.9530	0.317	0.168	0.099
Nahla [5]	Peak RP, N(t)	45.75429	807.8765	8020.848	43021.00	167739.3
	Peak time, t (s)	28.293	0.953	0.317	0.168	0.098

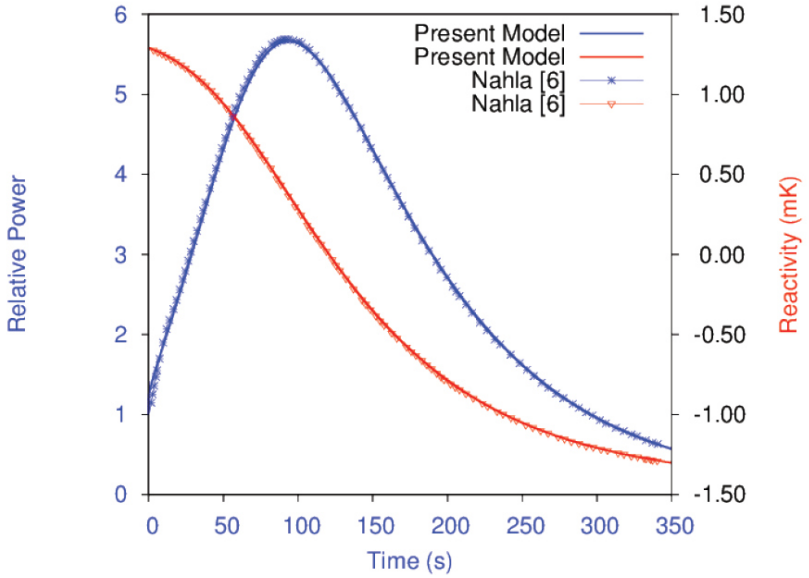

Figure 1: Temporal variation of relative power and reactivity for the initial reactivity 0.2$

Prediction for SCWR

The present case study demonstrates the validity of the model to analyze a supercritical water reactor (SCWR). The parameters λ_i and β_i are listed below in the table VII and the other input parameters for the present case study are Λ =0.00005 s and β=0.0065. Equations (1) and (2) are solved for six delayed neutron precursor groups assuming an equilibrium initial concentration of delayed neutron precursors. The relative power, N(t) obtained for 0.001 step reactivity with the model is compared with that obtained by Ambrosini [6] and the comparisons are shown in Fig 2. A good agreement among the comparing results is evident in the figure.

Table VII: Parameters for SCWR

Precursor group (i)	1	2	3	4	5	6
λ_i	0.0124	0.0305	0.111	0.301	1.14	3.01
β_i	0.000215	0.001424	0.001274	0.002568	0.000748	0.000273

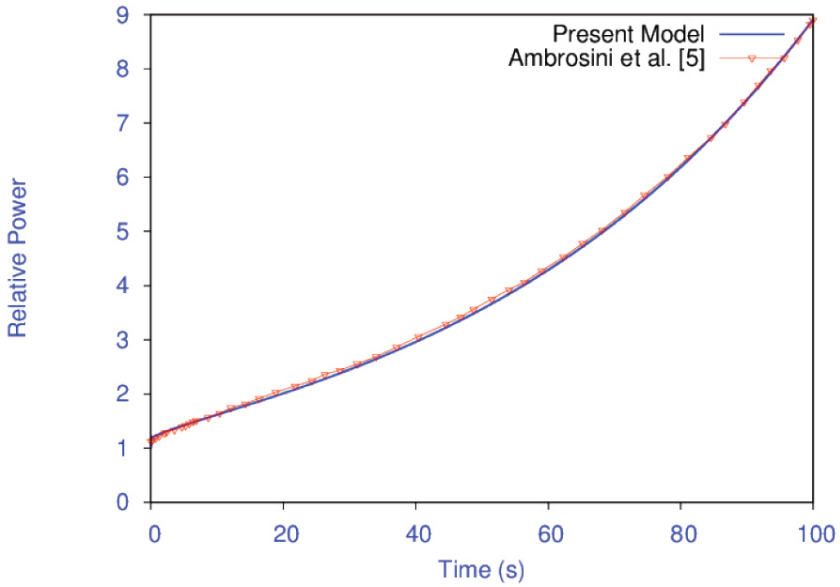

Figure 2: Temporal variation of relative power vs. time for step reactivity

Conclusions : The present model which is developed following the methodology proposed by Sanchez [1] is first tested against various available benchmark results. The capability of the model in taking into account the temperature dependent reactivity feedback effects and analyzing a SCWR are also verified during the study. In all the case study, a good agreement between the result obtained with the present model and the available results are demonstrated and therefore, it provides us the reasonable confidence to use the model for the nuclear coupled thermal hydraulics analysis of SCWR.

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