INTEGRATION AND DIFFEERENTIATION INVOLVING THE LAGUERRE POLYNOMIAL OF TWO VARIABLE L_n (x, y)

K.S.DHOUNSI, YASMEEN

Abstract : In this paper we obtain integration and partial differentiation involving the generalized associated Laguerre Polynomial of two variables $L_n^{(\alpha)}(x, y)$ which are is closelyrelated to generalized Lagguerre Polynomial of Dattoli et. al. These results provide useful extensions of well known results of Lagguere Polynomials $L_n(x)$

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Introduction: Two variable one index Laguerre polynomials have been given by Dattoli et.al. [1], [2] and [3].

Two variable one index Laguerre polynomials defined as

$$L_n(x, y) = n! \sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(n-r)!(r!)^2} \quad \dots (1.1)$$

L_n (x, y) are linked to the ordinary Laguerre polynomials L_n (x) by

$L_n(x, 1) = L_n(x)$, ... (1.2)

$L_n(x, y) = y^n L_n\left(\frac{x}{y}\right)$ .... (1.3)

A generalization of (1.1) provided by the following definition of the generalized associated Laguerre polyniomials

$$L_n^{(\alpha)}(x, y) = \sum_{r=0}^{n} \frac{(-1)^r (1+\alpha)_n y^{n-r} x^r}{(n-r)!\, r!\, (1+\alpha)_r}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r (\alpha+n)!\, y^{n-r} x^r}{r!\,(n-r)!\,(\alpha+r)!}, \quad (1.4)$$

and the generating function, we get

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x, y)\, t^n = (1-yt)^{-\alpha-1}\left(\frac{-xt}{1-yt}\right), \dots (1.5)$$

Now using expansion on R.H.S. in (1.5) and after some calculation, we get

$$x^n = \sum_{r=0}^{n} \frac{(-1)^r n!\,(1+\alpha)_n y^{n-r}}{(n-r)!\,(1+\alpha)_r} L_r^{(\alpha)}(x, y) \dots (1.6)$$

In this paper we shall give some basic relation and properties then obtain integral and differentiation involving the generalized associated Laguerre polynomials $L_n^{(\alpha)}$ (x, y)

Integral Involving Laguerre polynomials

I. To show

$$\int_0^{\infty} e^{-st} L_n(xt, y)dt = \frac{y^n}{s}\left(1-\frac{x}{sy}\right)^n \quad \dots (2.1)$$

Proof: Replace x by xt in L_n (x, y) and multiply by e^{-st} then integrating with respect to t with in Limits 0 to ∞

$$\int_0^{\infty} e^{-st} L_n(xt, y)dt = \int_0^{\infty} e^{-st} y^n$$

Put $\frac{xt}{y} = z$ in R.H.S. and using [5; P.216 (10)], we get required result (2.1)

II. To show

$$\int_0^{\infty} (x-t)^m L_n(t, y)\, dt = \frac{m!\, n!}{(m+n+1)!} L_n^{(m+1)}(x, y) \quad \dots (2.2)$$

Proof: $\int_0^{\infty} (x-t)^m L_n(t, y)\, dt$

$$= n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r}}{(n-r)!(r!)^2} \int_0^x (x-t)^m t^r\, dt$$

On putting t = xy, we get

$$= n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^{m+r+1}}{(n-r)!(r!)^2} \int_0^1 (1-u)^m u^r\, du$$

$$= n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^{m+r+1}\, m!}{(n-r)!(r!)(m+r\ \ 1)!}$$

$$= \frac{n!\, m!\, x^{m+1}}{(m+n+1)!} L_n^{(m+1)}(x, y)$$

which is a required result (2.2)

III If n is an odd in tiger then

$$\int_0^t L_n[x(t-x), y]\, dx = \frac{(-1)^n H_{2n+1}\left(t/2, y\right)}{2^{2n}\left(3/2\right)_n}$$

Proof: Taking L.H.S. of (2.3)

$$\int_0^t L_n(x(t-x), y\, dx$$

$$= \sum_{r=0}^{n} \frac{(-1)^r \; n! \, x^r \; (t-x)^r \; y^{n-r}}{(n-r)! \, (r!)^2} \, dx$$

put x = tu and using same procedure as [6; P. 153], we get

$$= \sum \frac{(-1)^{n-r} \; n! \, y^r \; t^{2(n-r)+1}}{r! \, (2n-2r+1)!}$$

$$= \frac{(-1)^n \; H_{2n+1}\left(t/2, y\right)}{2^{2n} \left(3/2\right)_n}$$

which is a required result (2.3)

IV. If n ≥ 1, then

$$\int_x^{\infty} e^{-y} \; [\mathbf{L}_n (y, z) - (z-1) \, \mathbf{L}_{n-1} (y, z)] \, dy$$

$= e^{-x} [\mathbf{L}_n (x, z) - z \, \mathbf{L}_{n-1} (x, z)]$ ...(2.4)

Proof: Taking L.H.S. =

$$\int_x^{\infty} e^{-y} \; [\mathbf{L}_n (y, z) - (z-1) \, \mathbf{L}_{n-1} (y, z)] \, dy$$

$= I_1 + I_2$ ...(2.5)

where $I_1 = \int_x^{\infty} e^{-y} \, \mathbf{L}_n (y, z) \, dy$

and $I_2 = - \int_x^{\infty} e^{-y} \, (z-1) \, \mathbf{L}_{n-1} (y, z) \, dy$

since $I_1 = e^{-x} \mathbf{L}_n (x, z) + \int_x^{\infty} e^{-y} \frac{\partial}{\partial y} \mathbf{L}_n (y, z) \, dy$ ...(2.6)

Now using differential recurrence relation for $\mathbf{L}_n (x, y)$

$$\frac{\partial}{\partial x} \mathbf{L}_n (x, y)$$

$= y \frac{\partial}{\partial x} \mathbf{L}_{n-1} (x, y) - \mathbf{L}_{n-1} (x, y)$; n ≥ 1 ...(2.7)

using (2.6) and (2.7) and after integrating we get $I_1 = e^{-x} [\mathbf{L}_n (x, z) - z \, \mathbf{L}_{n-1} (x, z)] - I_2$

Now using (2.5), then we get required result (2.4)

Partial differentiation of $\mathbf{L}_n (x,y)$:

Theorem – 3: If k be a positive integer then

$$\mathbf{L}_n^{(k)} (x, y) = (-1)^k \frac{\partial}{\partial x^k} \mathbf{L}_{n+k} (x, y)] \; ... \; (3.1)$$

Proof: By Definition of $\mathbf{L}_n (x, y)$

$$\mathbf{L}_{n+k} (x, y) = \sum_{r=0}^{n+k} \frac{(-1)^r \; (n+k)! \, x^r \; y^{n+k-r}}{(r!)^2 \; (n+k-r)!}$$

So that $(-1)^k \frac{\partial^k}{\partial x^k} \mathbf{L}_{n+k} (x, y)$

$$= \sum_{r=0}^{n+k} \frac{(-1)^{r+k} \; (n+k)! \, x^{r-k} \; y^{n+k-r}}{(r!) \, (n+k-r)! \, (r-k)!} \quad ... (3.2)$$

since $\frac{\partial^k}{\partial x^k} (x^r) = \frac{r!}{(r-k)!} x^{r-k}$, if the series starts from

r = 0; then $x^0 = 1$ and $\frac{\partial^k}{\partial x^k}$ can not be operated.

Thus $\frac{\partial^k}{\partial x^k}$ can be operated when power of x ≥ k

i.e. the series must start from r = k,

i.e. on changing the summation by substituting s = r – k in (3.2), we get

$(-1)^k \frac{\partial^k}{\partial x^k} \mathbf{L}_{n+k} (x, y)$

$$= \sum_{s=0}^{n} \frac{(-1)^{s+2k} \; (n+k)! \, x^s \; y^{n-s}}{(s+k)! \, (n-s)! \, s!} = \mathbf{L}_n^{(k)} (x, y)$$

which is a required result (3.1).

Special cases:

I. For x = y = 1, then (2.1) reduces to known result [5; P.216 (10)]

For y = 1, then (2.2) reduces to a known result [6; P.160 (9)]

For y = 1, then (2.3) reduces to a known result [4; P. 149 (5)]

For z = 1, then (2.4) reduces to a known result [4; P. 149 (4)]

II. For y = 1, then (3.1) reduces to a known result [4; P. 151 (7.5)]

Special cases from I and II are known formulae for integral and partial differentiation for ordinary Laguerre Polynomial $L_n(x)$.

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K.S.Dhounsi
Department of Mathematics,
Yasmeen
Government College, Kota-324001,
Rajasthan, India