TOTAL WEIGHT IRREGULARITY OF S- VALUED GRAPHS

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Abstract: Recently, we have studied the notion of Semiring valued graphs - called S-valued graphs. In our recent papers, we have studied the notion of Regularity on S-valued graphs, (a,k)-regular S-valued graphs and irregularity conditions on S-valued graphs. In this paper, we study the Total weight irregularity of S-valued graphs.

Key words: S-valued graphs, Regular S- valued graphs and (a,k)-regular S-valued graphs, Irregularity of S-valued graphs.

Introduction: In [1] and [2] the authors have discussed the notion of irregularity of graphs. Jonathan S. Golan [3] has introduced the notion of S −valued graph corresponding to a semiring S, where he considered a function $g: V \times V \to S$ such that $g(v_1, v_2) \neq \phi$. But nothing more has been dealt. In [5], we have introduced the notion of semiring valued graphs (simply called S −valued graphs). In [4] and [6], we have discussed the regularity and the degree regularity conditions on S −valued graphs. In this paper, we introduce the notion of total weight irregularity of S −valued graphs and study its properties.

Preliminaries: In this section, we recall some basic definitions from the theory of semirings, graphs and S −valued graphs that are needed in the sequel. Throughout our work, we consider Semirings S with zero element '0' and unit element '1'.

Definition 2.1. [1] Consider the graph G=(V ,E), with $|V| = n$, $|E| = m$.Let e=(u,v)∈ E . The imbalance of the edge e is defined as imb(e)=|deg(u) - deg(v)|.

Definition 2.2.[1] The irregularity of G is defined as irr(G) = $\sum_{e \in E} imb(e)$.

Definition 2.3.[2] The total irregularity of the graph G is defined as

irr$_T$(G) = $(\frac{1}{2})\sum_{u,v \in V} |\deg(u) - deg(v)|$.

Definition 2.4.[3] A semiring (S,+,·) is an algebraic system with a non-empty set S together with two binary operations + and · such that

1. (S, +, 0) is a monoid.
2. (S, ·) is a semigroup.
3. for all a,b,c ∈ S, a · (b+c) = a · b + a · c and (a + b) · c = a · c + b · c.
4. 0 · x = x ·0 = 0 ∀ x ∈ S.

Definition 2.5.[3] Let (S, +, ·) be a semiring. A canonical Pre-order ≲ in S is defined as follows : for a ,b ∈ S , a ≲ b if and only if, there exist a c ∈ S such that a + c = b.

Definition 2.6.[5] Let $G = (V, E \subset V \times V)$ be the underlying graph with $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a Semiring-valued graph (or a S-valued graph) G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$, where $\sigma: V \to S$ and $\psi: E \to S$ is defined to be

$$\psi(x,y) = \begin{cases} \min\{\sigma(x), \sigma(y)\} & if \;\; \sigma(x) \preccurlyeq \sigma(y) \\ & or \; \sigma(y) \preccurlyeq \sigma(x) \\ 0 & otherwise \end{cases}$$

for every unordered pair (x,y) of $E \subset V \times V$. We call σ, a S −vertex set and ψ, a S −edge set of S −valued graph G^S.

Definition 2.7. [4] If $\sigma(x) = a, \; \forall \, x \in V$ and for some $a \in S$ then the corresponding S −valued graph G^S is called a vertex regular S −valued graph.

Definition 2.8. [4] A S −valued graph G^S is said to be an edge regular S −valued graph if $\psi(x,y) = a$ for every $(x,y) \in E$ and for some $a \in S$.

Definition 2.9. [4] A S −valued graph G^S is said to be S −regular if it is both a vertex regular and an edge regular S −valued graph.

Definition 2.10. [6] Let G^S be a S −valued graph corresponding to an underlying graph G, and $a \in S$. G^S is said to be a $(a,k)-$ regular if it satisfies the following conditions:

1. The crisp graph G is k −regular.
2. $\sigma(v) = a$ for every $v \in V$.

Definition 2.11.[5] The Degree of the vertex v of the S-valued graph G^S is defined as deg$_S$(v)=($\sum_{(v_i, v) \in E} \psi((v_i, v), l)$

where l is the number of edges incident with v .

Total Weight Irregularity of S- valued Graphs: In this section, we introduce the notion of total weight irregularity condition on a S-valued graph by defining the notion of degree S-imbalance, weight S-imbalance. We will prove some simple results.

Definition 3.1. Consider a S-valued graph G^S = (V,E,σ, ψ) with |V| = n & |E| = m. Let e=(u, v) ∈ E and deg$_S$(u)=($\sum_{(u_i, u) \in E} \psi(u_i, u)$, $deg_G(u)$) , deg$_S$(v)=($\sum_{(v_j, v) \in E} \psi(v_j, v)$, $deg_G(v)$)

If $\sum_{(u_i,u) \in E} \psi(u_i, u) = \sum_{(v_j, v) \in E} \psi(v_j, v)$ then the degree S-imbalance of e is defined as degimb$_S$(e) = ($\sum_{(u_i,u) \in E} \psi(u_i, u)$, $deg_G(u) \sim deg_G(v)$) .ψ $\sum_{(v_j,v) \in E} \psi(v_j, v)$=a

⟹ degimb$_S$(e) =(c, $deg_G(u) \sim deg_G(v)$).

Since e=(u, v) ∈ E is arbitrary , degimb$_S$(e) exists for all edges.

Corollary 3.5. Let a∈S be idempotent in S and G^S be a vertex regular S-valued graph with S-vertex set {a} then $degimb_S$(e) =(a, $deg_G(u) \sim deg_G(v)$) for all e∈E.

Proof. Since every vertex regular S-valued graph is an edge regular S-valued graph , by above theorem $degimb_S$(e) =(a, $deg_G(u) \sim deg_G(v)$) for all e∈E.

Corollary 3.6. Let a∈S be idempotent in S and G^S be a regular S-valued graph with S-vertex set {a} then $degimb_S$(e) =(a, $deg_G(u) \sim deg_G(v)$) for all e∈E.

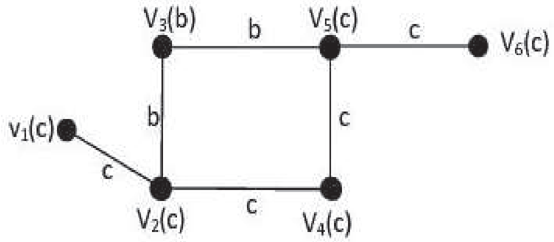
Proof. Since every S-regular graph is a S-vertex regular graph, by corollary 3.5., it holds.

Corollary 3.7. If G^S is a (a,k)-regular S-valued graph with a∈S as an idempotent element , then $degimb_S$(e) =(a, $deg_G(u) \sim deg_G(v)$) for all e∈E.

Proof: Since every(a,k)-regular S-valued graph is a S-regular graph, by corollary 3.6, it holds.

Definition 3.8. Degree S-irregularity of a S-valued graph G^S is defined as $degirr_S(G^S) = \sum_{e \in E} degimb_S(e)$.

Example 3.9. Consider the semiring in example 3.2. Let G^S be



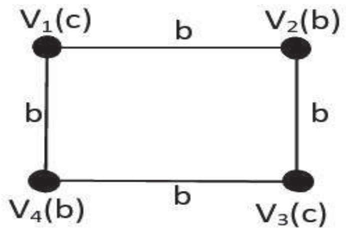
Clearly it is not a vertex regular hence it is not a regular or edge regular S-valued graph. and b is not an idempotent element but c is an idempotent element. Further S-edge set={b,c} and S-vertex set={b,c}.

Now, $deg_S(v_1)$=(c,1); $deg_S(v_2)$= (c, 3); $deg_S(v_3)$=(c,2); $deg_S(v_4)$=(c,2); $deg_S(v_5)$=(c,3); $deg_S(v_6)$=(c,1).

For an edge e=(v_1, v_2),
$degimb_S(e) = (c, 3 \sim 1) = (c, 2)$
For an edge e=(v_2, v_3) ,
$degimb_S(e) = (c, 1)$
For edge e=(v_2, v_4), $degimb_S(e) = (c, 1)$
For an edge e=(v_3, v_5) , $degimb_S(e) = (c, 1)$
For an edge e=(v_4, v_5) , $degimb_S(e) = (c, 1)$
For an edge e=(v_5, v_6) , $degimb_S(e) = (c, 2)$.
Therefore $degirr_S(G^S) = \sum_{e \in E} degimb_S(e)$ =**(c , 8).**

Remark 3.10 . From the above example we observe that the degree S-imbalance of all edges exists even if the S-valued graph G^S is not an edge regular, a vertex regular or not a S-regular with S-edge set is not a singleton set of an idempotent element of S. Therefore the above theorems & corollaries are not sufficient to the existence of degree S- imbalance of all edges.

Example 3.11. Consider the semiring in example 3.2. Let G^S be



Clearly it is an edge regular but not a vertex regular and b is not an idempotent element of S. But $deg_S(v_i)$= (c,2) , for all i= 1,2,3,4. Therefore $degimb_S(e) = (c, 0)$ for all e∈E.
⟹ $degirr_S(G^S)$=(c,0).

Remark 3.12 From the above example 3.11. , we observe that the existence of degree S-imbalance of all edges does not depend on the idempotent element.

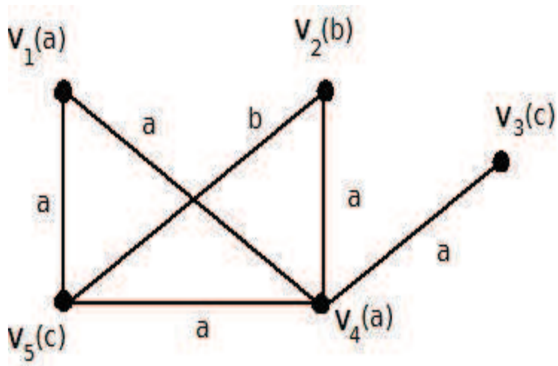
Theorem 3.13. The degree S-irregularity of a d_S-regular graph is (d,0) for some d ∈ S .

Proof: Let $G^S = (V,E,\sigma,\psi)$ be a d_S-regular graph . Therefore $deg_S(v_i)$= (a,k) , for all i and some a∈S. ⟹ $degimb_S(e) = (\sum_{(u_i,u) \in E} (\psi(u_i, u)$,
$deg_G(u) \sim deg_G(v))$
= (a+a+...+a, k~k)=(c,0) ,some c ∈ S.
Since e ∈ E is arbitrary, $degimb_S(e) = (c, 0)$ for all e ∈ E.
⟹ $degirr_S(G^S) = \sum_{e \in E} degimb_S(e)$ =(c+c+c+c+c+ . . . +c , 0)=(d,0).

Definition 3.14. Consider a S-valued graph $G^S = (V,E,\sigma,\psi)$ with |V| = n & |E| = m.
Let e=(u, v) ∈ E and $deg_S(u) = (\sum_{(u_i,u) \in E} \psi(u_i, u), deg_G(u))$,
$deg_S(v) = (\sum_{(v_j,v) \in E} \psi(v_j, v), deg_G(v))$.
If $deg_G(u) = deg_G(v)$, then the weight S-imbalance of e ∈ E is defined as
$wtimb_S(e) = (\min\{\sum_{(u_i,u) \in E} \psi(u_i, u), \sum_{(v_j,v) \in E} \psi(v_j, v)\}, deg_G(u))$.

Example 3.15. Consider the semiring and G^S as in example 3.2 and let e=(v_4, v_6). $deg_S(v_4)$=(b,3)and $deg_S(v_6)$=(c,3). Therefore we cannot find the $degimb_S(e)$ but $wtimb_S$(e)=(min{b,c}, 3)=(b,3).

Example 3.16. Consider the following S-valued graph G^S with the semiring in example 3.2.



Let e=$(u, v) \in E$ be arbitrary. Here $deg_G(u) \neq deg_G(v)$. Therefore we cannot find $wtimb_S(e)$ for all edges.

Theorem 3.17. If the underlying graph of a S-valued graph G^S is regular then $wtimb_S(e)$ exists for all e ∈ E.

Proof: Since the underlying graph G of G^S is regular,every vertex is of same degree. Especially , any two adjacent vertices of of an edge must have the same degree. Therefore by definition, $wtimb_S(e)$ exists for all e ∈ E.

Remark 3.18. $wtimb_S(e)$ exists only if the adjacent vertices of e are of same degree. Therefore we cannot find $wtimb_S(e)$ for all e even if the given S_valued graph is S-regular, S-vertex regular and S-edge regular.

Theorem 3.19. If G^S is a (a,k)-regular S-valued graph with a∈S as an idempotent element, then $wtimb_S(e)$=(a,k) for all e∈E.

Proof: Let $G^S = (V,E,\sigma,\psi)$ be a (a,k)-regular S-valued graph with a∈S as an idempotent element. ⟹It is a vertex regular S-valued graph and hence an edge regular S-valued graph. ⟹ $\psi(u, v)$ = a for any$(u, v) \in E$ and $deg_G(u)$=k for all $u \in V$ .By definition, $wtimb_S(e)$

= (min{ $\sum_{(u_i, u) \in E} \psi(u_i, u)$, $\sum_{(v_j, v) \in E} \psi(v_j, v)$ } , $deg_G(u)$) = (a,k) ($\because a + a = a$) , e∈E.

∴$wtimb_S(e)$=(a,k) for all e∈E.

Theorem3.20. In a d_S-regular S-valued graph, $wtimb_S(e)$=(a,k) for all e∈E and some a∈S is an idempotent element in S.

Proof: Since $deg_S(u)$ = (a,k) for some a∈S , for all u∈V and $deg_G(u)$= k for all u∈V in a d_S-regular S-valued graph,

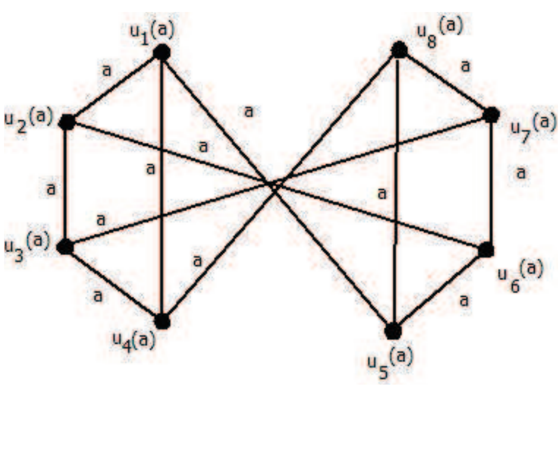
$wtimb_S(e)$= (min{ $\sum_{(u_i, u) \in E} \psi(u_i, u)$,
$\sum_{(v_j, v) \in E} \psi(v_j, v)$ } , $deg_G(u)$)
= (min{ a , a} , k) = (a,k).

∴$wtimb_S(e)$=(a,k) for all e∈E.

Definition 3.21. Weight S-irregularity of G^S is defined as

$wtirr_S(G^S) = \sum_{e \in E.} wtimb_S(e)$.

Example 3.22. Consider a semiring in example 3.2. Let G^S be



It is a (a,3) regular graph. Therefore for any e= $(u, v) \in E$,

$wtimb_S(e)$= (min{ $\sum_{(u_i, u) \in E} \psi(u_i, u)$,
$\sum_{(v_j, v) \in E} \psi(v_j, v)$ } , $deg_G(u)$)
= (min{ a+a+a+a , a+a+a+a} , 3)
= (min{b,b},3) = (b,3).

Since e is arbitrary , $wtimb_S(e)$=(b,3) for all e∈E.

∴ $wtirr_S(G^S) = \sum_{e \in E.} wtimb_S(e)$
= $\sum_{e \in E.}(b, 3)$ = (c, 24).

Definition 3.23. In general, S-imbalance of e=(u,v) ∈ E in G^S is defined as

$imb_S(e)$= (min{ $\sum_{(u_i, u) \in E} \psi(u_i, u)$,
$\sum_{(v_j, v) \in E} \psi(v_j, v)$ } , $imb_G(e)$).

Definition3.24. The S-irregularity of G^S is defined as $irr_S(G^S) = \sum_{e \in E} imb_S(e)$.

Definition 3.25. The Total S-irregularity of G^S is defined as $Tirr_S(G^S)$= ($min_{u \in V} \sigma(u)$,
$\frac{1}{2} \sum_{u,v \in V} |deg_G(u) - deg_G(v)|$).

Example 3. 26. Consider a semiring as in example 32. and G^S as in example 3.17.

Let e=(v_1, v_5). Then $imb_S(e)$= (min{ $\sum_{(v_1, u) \in E} \psi(v_1, u)$, $\sum_{(v_5, v) \in E} \psi(v_5, v)$ } , $imb_G(e)$).
= (min {b,c}, 1) = (b,1)

Similarly,if e=(v_5, v_2), then $imb_S(e)$= (min{ c , b} , $imb_G(e)$) = (b,1).

if e=(v_4, v_2) ,then $imb_S(e)$= (min{ b , b} , $imb_G(e)$) = (b,2).

if e=(v_3, v_4),then $imb_S(e)$= (min{ a , b} , $imb_G(e)$) = (b,3).

if e=(v_4, v_5), then $imb_S(e)$= (min{ b , b} , $imb_G(e)$) = (b,1).

if e=(v_1, v_4), then $imb_S(e)$= (min{ b , b} , $imb_G(e)$) = (b,2).

∴$irr_S(G^S) = \sum_{e \in E} imb_S(e)$ =(b+b+b+b+b+b , 8) = (c, 10).

Now $Tirr_S(G^S)$ = ($min_{u \in V} \sigma(u)$, $\frac{1}{2}$
$\sum_{u,v \in V} |deg_G(u) - deg_G(v)|$) =
(a, $\frac{1}{2}$ (0+1+2+1+1+2+1+3+2+1)=(a,$\frac{14}{2}$)
= (a,7)..

Remark 3.27. $degimb_S(e) = wtimb_S(e) = Imb_S(e)$ if G^S is (a,k) regular with a∈S as an idempotent element.

Conclusion: The further study is on irregularity of S-valued graphs discussed in this paper.

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