UNAVOIDABLE AND REPETITION FREE TWO DIMENSIONAL WORDS

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Abstract: Combinatorics on words is a field which appears in problems of computer science dealing with automata and formal languages Certain branches of mathematics constitute a major source of origin for formal language theory. The term "mathematical language theory" describes mathematical (algebraic) aspects of formal language theory – the main emphasis is on the mathematical theory rather than any applications. Hence our motivation comes from formal language theory and therefore the combinatorial aspect will be stressed more than algebraic aspect. Also formal languages with special combinatorial and structural properties which are exploited in information processing or information transmission. The theory has now developed into many directions and has generated a rapidly growing literature. In this paper we extended some special combinatorial one dimensional word properties into two dimensional arrays. This subtle distinction has dramatic consequences on the full paper. The goal of this paper is to give a brief survey on most important results of the theory of avoidable arrays or as its special case of repetition free arrays.

Key words: Complexity, Radiant palindrome ,Repetition free, Ultimately periodic and Unavoidable.

Introduction: The theory of words is profoundly connected to numerous different fields of mathematics and its applications and Chas (disorder).They might correspond to (one dimensional) materials having physical properties between crystals and glasses, and might be a good theoretical model of one- dimensional quasi crystals. In this paper we study we consider a problem area which has attracted quite a lot of attention in recent years, and which provides a plenty of extremely challenging combinatorial problems.

Basic Definitions and Notations: Let Σ be a finite alphabet and Σ^* be the free monoid generated by Σ. The elements of Σ^* are called words. The neutral element of Σ^* is called the empty word, is denoted by ξ and $\Sigma^+ = \Sigma^* - \{\xi\}$. The set of infinite words on Σ is denoted Σ^ω.

We now define Fibonacci word in terms of morphism

Let $B = \{a,b\}$ and $\phi: B^* \rightarrow B^*$ such that $\phi(a) = ab$, $\phi(b) = a$

The Fibonacci word is the fixed point of ϕ, $\phi^\omega(a)$ $(=\phi^\omega(b))$ and it is given by f = abaababa...

Let Σ^{**} be the set of all arrays over an alphabet Σ. The empty array is denoted by $\ominus$ Let $u = (u_{ij}) \in \Sigma^{**}$ $1 \le i \le m$, $1 \le j \le n$. The size of u is an ordered pair (m ,n) where m denotes the number of rows and n denotes the number of columns. Let $\Sigma^{++} = \Sigma^{**} - \ominus$. We adopt the convention that the bottom most row is the first row and the left most column is the first column.

An infinite array has an infinite number of rows and infinite number of columns. The collection of all infinite arrays over Σ is denoted by $\Sigma^{\omega\omega}$. The collection of all arrays with finite number of rows and infinite number of columns is denoted by $\Sigma^{*\omega}$ and the collection of all arrays with infinite number of rows and finite number of columns denoted by $\Sigma^{\omega *}$. It is clear that for row concatenation operation, the number of columns should be equal and for column concatenation operation, rows should be equal. These operations are denoted by the symbols $\ominus$ and $\oplus$ respectively and these operations are partial operations on Σ^{**}. u^ω denotes row concatenation of u with infinite number of u's with respect to row concatenation. Similarly, v^ω denotes the column concatenation of v with infinite number of v's with respect to column concatenation.

An array $w \in \Sigma^{\omega\omega}$ is said to be row ultimately periodic if there exist two arrays u and v in $\Sigma^{*\omega}$ such that

$w = u \ominus v^\omega$ An array $w \in \Sigma^{\omega\omega}$ is said to be column ultimately periodic if there exist two arrays u and v of $\Sigma^{\omega *}$ such that $w = u \oplus v^{\omega}$.

An array w is said to be ultimately periodic if it is both row ultimately periodic and column ultimately periodic. Also an ultimately periodic array is called radiant periodic if it is periodic along the diagonal. An array language L over the alphabet Σ is any subset of Σ^{**}. For each $m,n \ge 0$, we denote $\Sigma^{m \times n}$ the set of all arrays of size (m, n).for any finite array A, S(A) denotes the set of all its sub arrays , For any array language L, S(L) denotes the set of all sub arrays of L, that is $S(L)=\bigcup_{A \in L} S(A)$

Definition:2.2 For any $A \in \Sigma^{**}$, the sub array complexity of A is the map $g_A: N \times N \rightarrow N$ defined as

$g_A(m,n) = \text{card}(S(A) \cap \Sigma^{m \times n})$

where $g_A(m,n)$ counts the number of distinct sub arrays of A of size (p,q).

Definition :2.3 An infinite array u which is not ultimately periodic is said to be a sturmian array if $g_u(m,n) = m + n$ where $g_u(m,n)$ denotes the number of distinct sub factors of u, size (m,n)

. . b	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
a	B	.	.	.	.	.	.	.	.	.	.
a	A	b	.	.	.	.	.	.	.	.	.
b	A	a	b	.	.	.	.	.	.	.	.
a	B	a	a	b	.	.	.	.	.	.	.
b	A	b	a	a	b	.	.	.	.	.	.
a	B	a	b	a	a	b	.	.	.	.	.
a	A	b	a	b	a	a	b	.	.	.	.
b	A	a	b	a	b	a	a	b	.	.	.
a	B	a	a	b	a	b	a	a	b	.	.

Figure 1

The above array constructed from the Fibonacci word defined as the fixed point of the Fibonacci morphism f(a) = ab, f(b) = a. [5] . It is clear that its complexity satisfies

$g_\omega(m,n) = m + n$ for $m,n \geq 1$

Binary arrays satisfying the above complexity condition are so called infinite sturmian arrays. Our next simple result, [5] shows that the complexity of sturmian arrays is the smallest unbounded complexity. In particular, the Fibonacci array is an example of an array achieving this.

Unavoidable: Theorem:3.1 Let $w \in \Sigma^{\omega\omega}$ with $\|\Sigma\| \geq 2$.

If g_ω is not bounded then $g_\omega(m,n) \geq m + n$ for all $n \geq 1$ [6].

Corollary:3.2 Let $w \in \Sigma^{\omega\omega}$ with $\|\Sigma\| \geq 2$. Then ω is ultimately periodic if and only if g_ω is bounded.

Definition 3.3: A set $X \subseteq \Sigma^{**}$ is said to be unavoidable if there exist a pair of constants (i, j) such that each array $w \in \Sigma^{ij}$ contains an array of X as a factor.

Example 3.4: Consider example 3.1. The set $X = \{(2,2)(2,1)(1,2)\}$ is unavoidable over the Σ^{mn} for all m,n ≥ 2.

Lemma 3.5: Let $L \subseteq \Sigma^{**}$ be an infinte language. Then there exist an infinite array $w \in \Sigma^{\omega\omega}$ such that $F(w) \subseteq F(L)$. [6]

Definition:3.6: Let P be a property defined in the P is called an unavoidable regularity if the set $L = \{x \in \Sigma^{**} / x \text{ does not satisfy } P\}$ is finite. We say that P is an ideal property if P(x) implies P(lxm) for any $l, m \in \Sigma^{**}$. P is called avoidable if it is nor unavoidable. (i,e) there exist infinitely many words which do not satisfy P.

Proposition 3.7: Let P be an ideal property. The following statements are equivalent.

(i) there exist $w \in \Sigma^{\omega\omega}$ such that no factor of ω satisfies P.

(ii) P is avoidable

Proof: Clearly $(i) \Rightarrow (ii)$. Conversely, if P is avoidable then $L = \{x \in \Sigma^{**} / x \text{ does not satisfy } P\}$ is infinite. By lemma 3.6 there exists an infinite array $w \in \Sigma^{\omega\omega}$ such that $F(w) \subseteq F(L)$. Since P is an ideal property, it follows that L is closed by factors. Then L = F(L) and $F(w) \subseteq L$. Hence the proof.

Definition:3.8 Let u ∈ Σ**. A prefix of u is a rectangular sub array that contains one corner element, of u. A suffix is a rectangular sub array that contains the diagonally opposite corner. Notice that the choice of the corner in which to put the prefix is arbitrary. Because of the symmetry the prefix may be assigned to either the upper left or lower left corner of u.

Note: A proper prefix of an array $u = (u_{mn})$ is an array of size (p,q) where p<m-1 and q <n-1.

Theorem: 3.9 The number of proper prefixes of an array $u = (u_{mn})$ is (m-1)(n-1) for all m >2 and n >2

Proof: For finding out the number of proper prefixes (suffixes), we are omitting the right most column and the upper most row hence the total number is (m-1)(n-1).

Definition: 3.10 An array which coincides with its mirror image along the column (row) is called row (column) palindrome.

Definition: 3.11 An array which coincides with its mirror image along the radial diagonal is called radiant palindrome.

We observe that in row (column) palindrome every row(column) array is a palindrome word. Similarly in radiant palindrome we get a symmetric array. We also observe that the radiant palindrome is possible only for square arrays.

Definition:3.12 Prefixes which have radiant palindrome are called Special Prefixes.

4. Bordered Arrays: In recent years, some works have investigated border properties by looking at applications with DNA computing. In this section we focus on the border property of Sturmian arrays. The object of interest in this section is the finite arrays [2]. The phenomenon of border properties satisfying some combinatorial properties has been a key issue in the investigation of infinite arrays such as Sturmian array [8].

Definition 4.1: Let $U \in \Sigma^{**}$. A prefix of U is a rectangular sub array that contains one corner element, of U. A suffix is a rectangular sub array that contains the diagonally opposite corner. Notice that the choice of the corner in which to put the prefix is arbitrary. Because of the symmetry, the prefix may be assigned to either the upper left or lower left corner of U.

Definition 4.2: If a prefix and a suffix of an array are equal then the array is said to be bordered.

Example 4.3: A finite array with (3, 2) size border is depicted in the following.

a	b	a	a	a	b
b	b	a	a	b	b
a	b	b	b	a	a
b	b	b	b	a	b
a	a	b	b	a	b

Figure 2

The above array is also bordered by an array of size (1, 2) with respect to other diagonal also.

Definition 4.4: An array bordered by a prefix of size (2, 2) is said to be tile bordered.

Example 4.5: A tile bordered array is shown in the following.

a	b	a	a	b	b
b	b	a	a	b	b
b	b	b	b	a	a
b	b	b	b	a	b
b	b	b	b	a	b

Figure 3

It is interesting to note that in a Sturmian array the border property can be easily checked with the equality of the right boundary of the prefix and suffix of that array.

Definition 4.6: If the first and the last row (column) of an array are equal then the array is said to be row (column) bordered of width one.

Definition 4.7: If the first and the last r rows (columns) of an array are equal then the array is said to be row (column) bordered of width r.

Example 4.8 Let

$$A = \begin{matrix} a & a & b & b \\ b & a & a & b \\ a & a & a & a \\ a & a & b & b \end{matrix} \qquad B = \begin{matrix} a & b & b & a \\ b & a & a & b \\ a & a & a & a \\ a & a & b & a \end{matrix}$$

Figure 4

We can easily see that A is row bordered of width one and B is column bordered of width one.

Example 4.9: The following array is a column bordered array of width 3.

a	a	a	b	a	b	a	a	a
a	a	a	a	a	b	a	a	a
a	a	a	b	b	b	a	a	a
a	a	a	a	b	b	a	a	a
a	a	a	a	a	a	a	a	a
a	a	a	b	a	a	a	a	a
a	a	a	b	b	b	a	a	a
a	a	a	a	b	b	a	a	a
a	a	a	a	a	a	a	a	a

Figure 5

The above array is column bordered by single letter a and also has border of size (3, 3) in all the corners.

Conclusion: In this paper we have introduced the notion of a unavoidable, radiant palindrome, special prefix and obtain relationship between them.

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