ON INNER PRODUCT SPACES, ORTHOGONALITY, OTHONORMALITY

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Abstract: An Inner product space is a vector space with an additional structure called an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity. Inner products provide the means of defining orthogonality between vectors. It also allows some geometrical notions such as length of a vector or angle between two vectors.

Keywords: vector space, inner product space, normedspace, orthogonal set.

Introduction: The focus of this paper is on Inner product space,orthogonality of vectors,some properties and normed linear space.Besides these some theorems have been raised.

Definition: Inner product space: Let V be a vector space over a field F (field of scalars denoted by F is either the field of real numbers$\mathbb{R}$ or field of complex numbersC)together with a map $< ,> : V \times V \rightarrow F$ is called an inner product on V if it satisfies following axioms:

For every u,v,wε V, a,bε F

1 $<u,v> = \overline{< v,u >}$ (conjugate symmetry)

2 $<u,u>\geq 0$ and $<u,u> = 0$ iff $u = 0$ (positive definiteness)

3 $<au+bv,w> = a<u,v> +b<v,w>$ (linearity)

The vector space V with an inner product is called an inner product space.

When F =$\mathbb{R}$, conjugate symmetry reduces to symmetry,that is, $<u,v> = <v,u>$

While for F = C, $<u,v> = \overline{< v,u >}$

Now, conjugate symmetry and linearity implies $<u,bv+cw> = <bv+cw,u> =$

$b<v,u>+c<w,u>=b<v,u>+c<w,u> = b <u,v>+c<u,w>$

so$<u,av> = a<u,v>$

and$<u,v+w> = <u,v> +<u,w>$

thus an inner product is a sesquilinear form. Conjugate symmetry is also called Hermitian symmetry.

examples of inner product space

Example1: the vector space $\mathbb{R}^n$with dot product

$u.v = a_1b_1+a_2b_2+\_ \_ \_ +a_nb_n$

with $u = (a_1 ,a_2,\_ \_ \_,a_n)$, $v = (b_1, b_2,\_ \_ \_,b_n)$ ε$\mathbb{R}^n$is an inner product space. The vector space $\mathbb{R}^n$with this inner product is called Euclidean n-space.

Example 2 :The vector space C[a,b] of all real valued continuous function on closed interval [a,b] is an inner product space whose inner product is defined by, $<f,g> =\int_a^b f(t)g(t)dt$, $a \leq t \leq b$, f,gε C[a,b]

Definition : Normed linear space : A normed linear space is a vector space V over the field F and a non negative mapping $\|.\|$ on V , called the norm which satisfies the following axioms: for every u,vε V and a εF

1 $\|u\|\geq 0$ and $\|u\| = 0$ iff $u =0$

2 $\|au\|=|a|\|u\|$

3 $\|u+v\| \leq \|u\|+\|v\|$

Here $\|u\|$ is thought of as the length of u or distance of u from 0

By nonnegativity axiom of definition of inner product space, $\|u\| = \sqrt{<u,u>}$

Theorem: For any vectors u and v in an inner product space V, $|<u,v>| \leq \|u\|\|v\|$ or

$<u,v>^2 \leq <u,u><v,v>$

Proof: Let u and v be vectors in an inner product space V.In case v = 0then theorem is trivially true.So assume that $v \neq 0$.Let λεC be given by $\lambda = <u,v>/\|v\|^2$.

since$\|u-\lambda v\|^2 \geq 0 \Rightarrow <u-\lambda v,u-\lambda v>\geq 0 \Rightarrow < u-\lambda v,u> -\lambda<u-\lambda v,v>\geq 0$

$\Rightarrow<u,u> -\lambda<v,u>- \lambda<u,v> +\lambda\lambda<v,v>\geq 0$

$\Rightarrow\|u\|^2 - |<u,v>|^2 /\|v\|^2 - |<u,v>|^2 /\|v\|^2 + |<u,v>|^2 \|v\|^2 /\|v\|^2 \|v\|^2 \geq 0$

$\Rightarrow\|u\|^2 - |<u,v>|^2 /\|v\|^2 \geq 0 \Rightarrow |<u,v>| \leq \|u\|\|v\|$ and if the inequality holds as an equality, then $\|u-\lambda v\| = 0\Rightarrow u-\lambda v = 0$

Thus u and v are linearly independent and conversely.

Orthogonality:

Definition: Let V be an inner product space .Two vectors u,v are said to be orthogonal if $<u,v> = 0$

Example: for inner product space C[-π,π], the functions sint ,cost are orthogonal as

$<\sin t,\cos t>=\int_{-\pi}^{\pi} \sin t \cos t\, dt =1/2\int_{-\pi}^{\pi} \sin 2t\, dt= -1/4[\cos 2\pi-\cos(-2\pi)] = -1/4[1-1] = 0$

orthogonal complement: Let S be a non empty subset of inner product space V. The orthogonal complement of S, denoted by $S^\perp$ .consists of those vectors in V that are orthogonal to every vector u εS,that is, $S^\perp = \{v \in V / <v,u> = 0$ for every $u \in S\}$

In particular, for a given vector u in V, $u^\perp=\{ v \in V / <v,u> = 0\}$,that is, $u^\perp$ consists of all vectors in V that are orthogonal to vector u.

Theorem: Let S be a subset of a vector space V then $S^\perp$ is a subspace of V

Theorem: Let S be a subset of inner product space V,then every vector of $S^\perp$ is orthogonal to every vector of span (s)

Proof: let $S = \{u_1,u_2,\_ \_ \_,u_r\}$ for any u ε span (S) we have

$u=a_1 u_1+a_2 u_2+\_ \_ \_ \_ \_ +a_r u_r$ where $a_i \varepsilon$ F and for any v $\varepsilon S^{\perp}$,we have <v,w> = 0 for every wεS <u,v>= $<\sum_{i=1}^{r} a_i u_i , v> = \sum_{i=1}^{r} a_i <u_i , v>$ = 0

thus,<u,v> = 0 for all u εspan (S),v $\varepsilon S^{\perp}$

Example: Let A be an m×matrix then null A and Row A are orthogonal complements of each other in $\mathbb{R}^n$

Otrthonormal Set:

Definiton : Let V be an inner product space .Consider a set $S=\{u_1,u_2,\_ \_ \_,u_r\}$ of non zero vectors in V then S is called orthonormal set if S is orthogonal and each vector in S has unit length,that is, $<u_i,u_j> = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{for } i=j \end{cases}$

Theorem: (Pythagoras):Suppose $\{u_1,u_2,\_ \_ \_,u_r\}$ be orthogonal set of vectors then

$\| u_1+ u_2+ \_ \_ \_ +u_r\|^2 =\| u_1\|^2+\| u_2\|^2+ \_ \_ \_\|u_r\|^2$

Proof: for simplicity assume that r=2,nowu_1, $u_2$are orthogonal so $< u_1, u_2> = 0$ and

$\| u_1+ u_2\|^2 =< u_1+ u_2, u_1+ u_2>= < u_1, u_1>+< u_1, u_2>+< u_2, u_1>+< u_2, u_2>$

$=\| u_1\|^2+\| u_2\|^2 +2< u_1, u_2> =\| u_1\|^2+\| u_2\|^2$

Theorem: Let W be a subspace of finite dimensional inner product space Vthen

Then V is direct sum of W and $W^{\perp}$

Proof:Since W and $W^{\perp}$ are subspaces of finite dimensional inner product space V thus dim(W) <∞ and $\dim(W^{\perp})<\infty$ Let dim (V)= n . if dim(W) = r then let $\{u_1,u_2,\_ \_ \_,u_r\}$ be basis of W and let $\dim(W^{\perp})$=s then let $\{u_{r+1},u_{r+2},\_ \_ \_,u_n\}$ be basis of $W^{\perp}$ now from theBasis of subspaces we form the basis of V,so we have $\{ u_1,u_2,\_ \_ \_,u_r,u_r\ u_{r+1},u_{r+2}\ ,\_ \_ \_,u_n\}$ is basis of V.obviously $W+ W^{\perp}\subset V$ to show the converse we consider u ε V therefore u is linear combination of vectors of basis of V,that is , $u= a_1 u_1+a_2 u_2+\_ \_ \_ +a_r u_r+a_{r+1} u_{r+1} + \_ \_ \_ +a_n u_n = \sum_{i=1}^{r} a_i u_i + \sum_{j=r+1}^{n} a_j u_j$ $\varepsilon W + W^{\perp}$. thus $V = W+ W^{\perp}$ .Now proof is complete by showing that $W\cap W^{\perp} = \{0\}$.for this let u ε $W\cap W^{\perp}\Rightarrow$ u ε w and u $\varepsilon W^{\perp}$.now u $\varepsilon W^{\perp}$ implies <u,w>=0 for every w εW in particular <u,u> = 0 ⇒ u=0 thus $W\cap W^{\perp}= \{0\}$

3.4 Theorem: Let $\{u_1,u_2,\_ \_ \_,u_n\}$ be an orthogonal basis of V then for any v ε V,

$v=<v, u_1> u_1/< u_1, u_1>+<v, u_2> u_2/< u_2, u_2>+ \_ \_ \_+<v,u_n> u_n/< u_n, u_n>$

Proof:since vεVand $\{ u_1,u_2,\_ \_ \_,u_n\}$ be orthogonal basis of Vtherefore v is linear combination of vectors of V we write $v= a_1 u_1+ a_2 u_2+\_ \_ \_ +a_n u_n$ where$a_i \varepsilon$ F

Now $<v,u_j>= <\sum_{i=1}^{n} a_i u_i , u_j> = \sum_{i=1}^{n} a_i <u_i,u_j>=a_j<u_j,u_j>\Rightarrow a_j =<v,u_j>/\|u_j\|^2$

Thus $v= \sum_{i=1}^{n} a_i u_i = \sum_{i=1}^{n} <v,u_i>u_i/\|u_i\|^2$

Gram Schmidt theorem :Let V be an inner product space and $\{u_1,u_2,\_ \_ \_,u_n\}$ is basis of V first we construct an orthogonal basis $\{v_1,v_2,\_ \_ \_,v_n\}$ as follows,

Set $v_1 =u_1,v_2 = u_2 -< u_2, v_1> v_1/< v_1, v_1>$

If v_2= 0 then u_2 is linear combination of $v_1 = u_1$therfore$\{ u_1, u_2\}$ is linearly dependent,which is not possible $\Rightarrow v_2\neq 0$ now $< v_2, v_1>=$

$<u_2 -< u_2, v_1> v_1/< v_1, v_1>,v_1> =< u_2, v_1>- < u_2, v_1>< v_1, v_1>/< v_1, v_1>=0$

Similarly, $v_3 = u_3-< u_3, v_1> v_1/< v_1, v_1>-< u_3, v_2> v_2/< v_2, v_2>$ and on same steps $< v_3, v_2> = 0$,$< v_3, v_1> = 0$- - - - - - -

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$v_n= u_n- < u_n,v_1> v_1/< v_1, v_1>-< u_n, v_2> v_2/< v_2, v_2>$ - - -$< u_n, v_{n-1}> v_{n-1}/< v_{n-1}, v_{n-1}>$

thus $\{ v_1,v_2,\_ \_ \_,v_n\}$ is orthogonal basis of V .normalizing each v_i, that is, $w_i = v_i/\| v_i\|$

$\Rightarrow<w_i,w_i>=1$ and $<w_i,w_j> = 0$ for i≠ j thus$\{ w_1, w_2,_ _ _ ,w_n\}$ is orthonormal basis.

IsometryOn Vector Space

Definition: Let V be an n-dimensional vector space. A linear transformation

T: V→ V is called an isometry if for any v ε V $\|T(v)\| = \|v\|$

Theorem :A linear transformation T : V → V is an isometryiff T preserves inner product that is for u,vε V ,<T(u),T(v)> = <u,v>

Proof: first let T is an isometry thus $\|T(u+v)\|= \|u+v\|$

$\Rightarrow\|T(u+v)\|^2=\|u+v\|^2\Rightarrow<T(u+v),T(u+v)>=<u+v,u+v>$

$\Rightarrow<T(u)+T(v),T(u)+T(v)>=<u+v,u+v>$

$\Rightarrow\|T(u)\|^2 +\|T(v)\|^2+2<T(u),T(v)> = =\|u\|^2+\|v\|^2 +2<u,v>$

$\Rightarrow\|u\|^2+\|v\|^2+2<T(u),T(v)>=\|u\|^2+\|v\|^2+2<u,v>\Rightarrow<T(u),T(v)>=<u,v>$

Conversely Let T preserves inner product ,that is, <T(u),T(v)>=<u,v>

Consider $\|T(v)\|^2 =<T(v),T(v)>=<v,v>=\|v\|^2\Rightarrow\|T(v)\| = \|v\|$ thus T is an isometry

Theorem: Let T: V → V be an isometry then T is injective

Proof: it suffices to prove that kernel is trivial for this let v εker(T) ⇒ T(v)=0

$\Rightarrow<T(v),T(v)>=0 \Rightarrow\|T(v)\| =0\Rightarrow\|v\|= 0 \Rightarrow v=0\Rightarrow \ker (T)=0$

Conclusion: Any of the axioms of an inner product may be weakened, yielding generalized nations. The generalizations that are closest to inner product occur where bilinearity& conjugate symmetry are retained but positive definiteness is weakened.

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