ALLIED REGULAR SPACES VIA α-OPEN SETS

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Abstract: The purpose of this paper is to introduce and study new classes of weaker regular spaces viz. α-p-regular and α-s-regular spaces. We obtain new characterizations and establish various preservation theorems of such spaces.

Keywords: preopen, semi-open and α-open sets, α-p-regular and α-s-regular spaces etc.
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Introduction: The notions of semi-open sets were introduced and studied by Levine [6] in 1963. Njastad [11] and Mashhour et al [9] introduced and studied the concept of α-open (originally called α-sets) and α-closed sets respectively in topological spaces. In 1982, A.S. Mashhour et al.[10] have defined the notions of preopen sets and precontinuity in topology . Since then many topologists have utilized these concepts to the various notions of subsets, weak separation axioms, weak regularity. M.K.Singal [16], Maheshwari [7]. El. Deeb [4], Navalgi [13] and Park and Park[15] defined and studied almost regular, s-regular, p-regular, α-regular and sp-regular. In this paper we introduce and study new classes of weaker regular spaces viz. α-p-regular and α-s-regular spaces. We obtain some new characterizations and establish various preservation theorems of such spaces.

Preliminaries: Throughout this paper X, Y will denote topological spaces on which spaces no separation axioms are assumed unless explicitly stated. Let f: X→Y represents a single valued function. Let A be a subset of X. The closure and interior of A are denoted by Cl(A) and Int(A) respectively. The following definitions and results are useful in the sequel.

Definition 2.1 : A subset A of a space X is called

1. semi – open [6]- if $A \subset Cl(Int(A))$.
2. α-open [11]- if $A \subset Int(Cl(Int(A)))$.
3. preopen [10]- if $A \subset Int(Cl(A))$.

The complement of semi-open (resp. α-open, preopen) set is said to be semi-closed (resp. α-closed, preclosed). The collection of all closed (resp. preopen semi–open, α-open) subsets of X will be denoted by C(X) (resp. PO(X), SO (X), αO(X)).

Definition 2.2 : The α-interior [2] of A, denoted by αInt A and is defined as the union of all α-open sets which are contained in A.

Definition 2.3 : The α-closure [9] of a subset A of X is the inters of all α-closed sets that contain A and is denoted by αCl(A).

Definition 2.4 : A space X is said to beα-regular [ ] if for each closed set F and a point $x \in X-F$, there exist disjoint α - open sets U, V ∈ X such that $F \subset U$ and $x \in V$.

Definition 2.5 : A space X is said to be strongly regular [12] if for each preclosed set F and each point $x \in X-F$, there exist disjoint preopen sets U and Vsuch that $x \in U$ and $F \subset V$.

Definition 2.6: A function f: X→Y is said to be strongly precontinuous [1] if $f^{-1}(V)$ is pre open in X for every semi–open set V of Y.

Definition 2.7: A function f: X→Y is said to be strongly semi-continuous [3] if the inverse image of each semi-open set of Y is open set in X.

Definition 2.8 : A function f : X→Y is said to be α-irresolute [8] function, if $f^{-1}(V)$ is α- open set in X for every α - open subset V of Y.

Definition 2.9 : A function f : X →Y is said to be α-open[9] function if the image of each open set of X is α-open in Y.

Definition 2.10: A function f:X →Y is called strongly α-closed [17] if the image of each α-closed set in X is α- closed set in Y.

Definition 2.11: A function f:X →Y is called αg-closed [4] if f(F) αg-closed in Y for every closed set F of X.

Definition 2.12: A function f:X →Y is called pre-semi-closed [5] if f(F) semi-closed in Y for every semi-closed set F of X.

Regularity Axioms Via α-Open Sets: We introduce the following;

Definition 3.1: A space X is said to be α-p-regular if for each preclosed set F of X and each point $x \in X-F$, there exist disjoint α - open sets U, V such that $F \subset U$ and $x \in V$.

Clearly, we have the following implications

(i) every α - p-regular space is strongly – regular space but the converse is not true.

(ii) every α -regular space is α-p-regular space but the converse is not true

The following examples show the above converses, which are not true.

Example 3.2: Let X = {a,b,c} τ = {X, ϕ,{a}, {b}, {a, b}}. αO(X) = { X, ϕ,{a}, {b}, {a, b}, {a,c }} PO(X) = { X, ϕ,{a}, {b}, {a, b}}. Clearly, X is strongly – regular space but not α - p-regular space.

Example 3.3: Let X = {a,b,c} τ = {X, ϕ,{a}}. αO(X) = { X, ϕ,{a},{a, b},{a,c }}. Clearly, X is α - p-regular space but not α - regular space.

Now we characterize α-p- regular spaces in the following.

Theorem 3.4: In a topological space X, the following conditions are equivalent:

(i) X is α- p- regular

(ii) for each preclosed set F and each point x∈ X-F, there exist U∈αO(X) and an αg-open set V such that x∈U, F ⊂V and U∩ V=ϕ.

(iii) for each subset A of X and each preclosed set F such that A∩ F= ϕ, there exist U∈αO(X) and an αg-open set V such that A∩ U= ϕ, F ⊂ V and U∩ V=ϕ.

Proof: (i) ⇒ (ii) The proof is obvious since every α-open set is an αg-open.

(ii) ⇒ (ii) Let A be a subset of X and F a preclosed set X such that A∩ F= ϕ. For a point x∈A, x∈ X-F and hence there exists U∈αO(X) and an αg-open set V such that x∈U, F ⊂V and U∩ V=ϕ.

(iii) ⇒ (i) Let F be any preclosed set of X and x∈ X-F. Then {x}∩ F= ϕ and there exist U∈αO(X) and an αg-open set W such that x∈U, F ⊂W and U∩ W=ϕ. Put V=αInt(W) then by remark [14] 1.2.24, we have F ⊂V, V∈αO(X) and U∩ V=ϕ. There X is α-p-regular.

Theorem 3.5: In a topological space X, the following statement are equivalent:

(i) X is α- p- regular

(ii) For each preopen set U of X containing x, there exists α-open set V such that x∈V⊂ αCl(V) ⊂ U.

Proof: (i) ⇒ (ii) Let x∈X and U be preopen set of X containing x. Then X-U is a preclosed and x ∉ X-U. By (i) there exist disjoint α-open sets V and W such that x∈V and X-U⊂W. That is X-W⊂U. X-W is α-closed. Therefore αCl(X-B)⊂U. Since V∩ W= ϕ , V⊂X-W. Therefore, x∈V⊂ αCl(V) ⊂ αCl(X-B) ⊂U. Hence, x∈V⊂ αCl(V) ⊂ U.

(ii) ⇒ (i) Let F be any preclosed set and x ∉ F. This implies that X-F is a preopen set containing x. By (ii), there exists an α-open set V such that x∈V⊂ αCl(V) ⊂ X- F . Let U = X –αCl(V) then U is an α-open set such that F ⊂ U, x∈V and U∩ V=ϕ. Thus, there exists disjoint α-open sets U and V such that x∈V, F ⊂ U. Therefore, X is α-p-regular. We introduce the following;

Definition 3.6: A space X is said to be α-s-regular if for each semiclosed set F of X and each point x∈ X-F, there exist disjoint α - open sets U, V such that F⊂ U and x ∈ V.

Clearly, we have the following implications

(i) Every α - s-regular space is semi –regular space but the converse is not true.

(ii) Every α -regular space is α-s-regular space but the converse is not true

The following examples show the above converses, which are not true.

Example 3.7: Let X = {a,b,c} τ = {X, ϕ,{a}, {b, c}}. αO(X) = { X, ϕ,{a},{a, b},{a,c }, {b,c}}, SO(X) = { X, ϕ,{a} {b, c}}. Clearly, X is semi – regular space but not α - s-regular space.

Example 3.8: Let X = {a,b,c} τ = {X, ϕ,{a}, {a, b}}. αO(X) = { X, ϕ,{a}, {a, b},{a,c }}. Clearly, X is α - s-regular space but not α -regular space.

Now we characterize α-s- regular spaces in the following.

Theorem 3.9: In a topological space X, the following conditions are equivalent:

(i) X is α- s- regular

(ii) for each semi-closed set F and each point x∈ X-F, there exist U∈αO(X) and an αg-open set V such that x∈U,F ⊂V and U∩ V=ϕ.

(iii) for each subset A of X and each semi-closed set F such that A∩ F= ϕ, there exist U∈αO(X) and an αg-open set V such that A∩ U≠ ϕ, F ⊂V and U∩ V=ϕ.

Proof: (i) ⇒ (ii) The proof is obvious since every α-open set is αg-open.

(ii) ⇒ (iii) Let A be a subset of X and F a semi-closed set X such that A∩ F= ϕ. For a point x∈A, x∈ X-F and hence there exists U∈αO(X) and an αg-open set V such that x∈U, F ⊂V and U∩ V=ϕ.

(iv) ⇒ (i) Let F be any semi-closed set of X and x∈ X-F. Then {x}∩ F= ϕ and there exist U∈αO(X) and an αg-open set W such that x∈U, F ⊂W and U∩ W=ϕ. Put V=αInt(W) then by remark [14] 1.2.24. we have F ⊂V, V∈αO(X) and U∩ V=ϕ. There X is α-s-regular.

Theorem 3.10: In a topological space X, the following statement are equivalent:

(i) X is α- s- regular

(ii) for each semi-open set U of X containing x, there exists α-open set V such that x∈V⊂ αCl(V) ⊂ U. **Proof:** (i) ⇒ (ii) Let x∈X and U be semi-open set of X containing x. Then X-U is a semi-closed and x ∉ X-U. By (i) there exist disjoint α-open sets V and W such that x∈V and X-U⊂W. That is X-W⊂U. X-W is α-closed. Therefore αCl(X-B)⊂U. Since V∩ W= ϕ , V⊂X-W. Therefore, x∈V⊂ αCl(V) ⊂ αCl(X-B) ⊂U. Hence, x∈V⊂ αCl(V) ⊂ U.

(ii) ⇒ (i) Let F be any semi-closed set and x ∉ F. This implies that X-F is a semi-open set containing x. By (ii), there exists an α-open set V such that x∈V⊂ αCl(V) ⊂ X- F . Let U = X –αCl(V) then U is an α-open set such that F ⊂ U,,x∈V and U∩ V=ϕ. Thus, there exists disjoint α-open sets U and V such that x∈V, F ⊂ U. Therefore, X is α-s-regular.

Some preserving theorems on regularity axioms via α-open sets.

Theorem 3.11: If f: X → Y be strongly pre-continuous surjection on with compact inverses. If X is regular then Y is α-p-regular.

Proof: Let F be a preclosed set V and y∉ F. Then f^{-1}(F) is closed in X and f^{-1}(y) is a compact, Moreover, f^{-1}(F) and f^{-1}(y) are disjoint in the regular space X. Hence, there exist open sets U and V such that f^{-1}(y) ⊂ U and f^{-1}(F) ⊂ V . Since f is α-closed by the Theorem 3.5 there exists A,B∈ αO(Y) such that y∈A, F ⊂ B, f^{-1}(A) ⊂ U and f^{-1}(B) ⊂ V. Since, U∩ V=ϕ and f is surjection we have A∩ B=ϕ. This shows that Y is α-p-regular.

Theorem 3.12: If f: X → Y be strongly pre-continuous α-open, αg-closed surjection and X is regular then Y is α-p-regular.

Proof: Let y∈Y and V be an preopen set of Y containing y. Let x be a point of X such that y = f(x). By the regularity of X, there exists an open set U of X such that x∈U⊂ αCl(U) ⊂ f^{-1}(V). Then, we have y∈f(U) ⊂ f(αCl(U)) ⊂ V. Since f is α-open and αg-closed, f(U) ∈ αO(Y) and f(Cl(U)) is αg-closed in Y. Threfore, we obtain y∈f(U)⊂ αCl(f(U)) ⊂ αCl(f(ClU)⊂ V. It follows from Theorem 3.5 That is, Y is α-p-regular.

Theorem 3.13: If f: X → Y be strongly semi-continuous strongly α-closed bijective function and X is α-s- regular space then Y is α-s-regular.

Proof: Let y∈Y and V be semi-open set containing y of Y. Let x be a point of X such that y = f(x). Since, f is strongly semi-continuous, f^{-1}(V) is open in X. By assumptions and Theorem 3.10 There exists an α-open set U such that x∈U⊂ αCl(U) ⊂ f^{-1}(V). Then, we have y∈f(U)⊂ f(αCl(U)) ⊂ V. we know that αCl(U) is α-closed. Since f is strongly α-closed, αCl(U)) is closed. Therefore, we have αClf(αCl(U)) ⊂V. This implies that y∈f(U)⊂ αCl(f(U)) ⊂ V. Now U is α-open implies X-U is α-closed in X. Since f is strongly α-closed function, f(X-U) is closed in Y. That is Y-f(U) is closed in Y. This implies f(U) is open in Y. Since every open set is α-open set. Therefore, f(U) is α-open in Y. Thus, every point of y of Y and every semi-open set V containing y there exists an α-open set f(U) such that y∈f(U)⊂ αCl(f(U))⊂ V. It follows from Theorem 3.10 That is, Y is α-s-regular. **Theorem 3.14:** If f: X → Y is an α-irresolute pre semiclosed injection and Y is α-s-regular ,then X is α-s-regular. **Proof:** Let F be a semi-closed set of X and x∈X- F. Since, f is pre semi-closed, f(F) is semi-closed set in Y and f(x) ∈ Y-f(F). Since Y is α-s-regular, there exist disjoint α-open sets U and V such that f(x)∈U and f(F) ⊂ V. Therefore, we obtain x∈f^{-1}(U) and F ⊂ f^{-1}(V) and f^{-1}(U)∩ f^{-1}(V)=ϕ. Since f is α-irresolute f^{-1}(U) and f^{-1}(V) are disjoint α-open sets in X. This shows that X is α-s-regular.

Theorem 3.15: If f: X → Y be strongly semi-continuous α-open, αg-closed surjection and X is regular then Y is α-s-regular.

Proof: Let y∈Y and V be semi-open set y of Y. Let x be a point of X such that y = f(x). By the regularity of X, there exists an open set U of X such that x∈U⊂ αCl(U) ⊂ f^{-1}(V). Then, we have y∈f(U)⊂ f(αCl(U)) ⊂ V. Since f is α-open and αg-closed, f(U) ∈ αO(Y) and f(Cl(U)) is αg-closed in Y. Threfore, we obtain y∈f(U)⊂ αCl(f(U))⊂ αCl(f(Cl(U))⊂ V. It follows from Theorem 3.10 That is, Y is α-s-regular.

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