(1, α)-DERIVATIONS ON LIE IDEALS IN σ- PRIME NEAR-RINGS

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Abstract: Let N be a 2 - torsion free σ- prime near-ring and 1, $\alpha \in$ Aut N. Let d be a nonzero (1, α)-derivation which commutes with σ, and L be a nonzeroσ- Lie ideal, then N is commutative if any one of the following conditions hold. (i) $[d(u), u]_{1,\alpha} = 0$. (ii) $d(u)u = \alpha(u)d(u)$ (iii) $d(u^2) = \pm\, \alpha(u^2)$ (iv) $d(u^2) = 2\, d(u)\alpha(u)$, for all $u \in L$.

Keywords: Center, Near rings, (1, α,)-derivation,σ- Lie ideal, σ - prime near-ring,.

Introduction: During the last years, the study of commutativity of prime rings or prime near-rings was one of the most important subjects in the researches of algebraists. In this direction, Bell and Mason [3] initialized this study using the notion of derivation defined in a prime-ring. Argac [1] continued on a same line, he introduced the notion of two sided α-derivation.

Bell, Boua and Oukhtite generalized some results known in this field involving the semigroup ideal instead of near-ring and generalized derivation instead of the usual derivation. Hence, it should be interesting to study the commutativity of near-ring N admitting some conditions.

Boua and Kamal [9] concept of two-sided α - generalized derivation in prime near-rings as it was outlined by the author N. Argac [1]. Thereafter, they generalized the same results proved by many authors (see [2] and [8]) in the case of derivations, semiderivations and generalized derivations. Furthermore, they gave examples to show the restrictions given in the hypothesis of various results are not expendable.

Let N be an associative near-ring with center Z. We will write for all $x, y \in N$, $[x, y] = xy - yx$ for a commutator and the Jordan product by $x \circ y = xy + yx$. N is said to be 2-torsion free if whenever $2x = 0$, with $x \in N$, then $x = 0$. N is prime if $aNb = 0$ implies that $a = 0$ or $b = 0$ for all a and b in N. If σ is an involution in N, then N is said to be σ- prime if $aNb = aN\sigma(b) = 0$ implies that $a = 0$ or $b = 0$. It is obvious that every prime ring equipped with an involution σ is also σ- prime, but the converse need not be true in general. An ideal I of N is said to be a σ ideal i.e. I is invariant under σ, i.e., $\sigma(I) = I$. In all that follows $Sa_\sigma(N)$ will denote the set of symmetric and skew symmetric elements of N, i.e., $Sa_\sigma(N) = \{x \in N \,/\, \sigma(x) = \pm x\}$.

A Lie ideal of N is an additive subgroup U of N satisfying U of N satisfying $[u, n] \in U$ for all $u \in U$. Moreover if $\sigma(U) = U$, then U is called a σ- Lie ideal. Let I be a ring of integers. Set

$N = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} / a, b, d \in I\right\}$ and

$L = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} / b \in I\right\}$.

We define the following maps on N : $\sigma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} -d & b \\ 0 & a \end{pmatrix}$. Then L is a nonzero σ- Lie ideal of N.

An additive mapping $d : N \to N$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in N$. For a fixed $a \in N$, the mapping $I_a : N \to N$ given by $I_a(x) = [a, x]$ is a derivation which is said to be a inner derivation. Let α and β be any two automorphism of N. An additive mapping $d : N \to N$ is called a (α, β)-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in N$ and is said to be(1,α)-derivation if $d(xy) = d(x)y + \alpha(x)d(y)$ holds for all $x, y \in N$.

Let S be a nonempty subset of N. A mapping d from N to N is called centralizing on S if $[d(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[d(x), x] = 0$ for all $x \in S$. The study of centralizing and commuting mapping was initiated by Posner in [12]. (Posner's second theorem). Several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations which are centralizing and commuting on appropriate subsets of R (see eg., [4, 6, 5, 10] and references there in). Recently, oukhtite et a1.[11, Theorem 1] proved Posner's second theorem for 2-torsion free rings with involution.

Our attempt in this paper is to prove commutativity conditions for a 2-torsion free near-ring N with (1,α)-derivations on σ - Lie ideals. We follow the procedure adapted by Agrac [1] to obtain the results in this paper. N is commutative if any one of the following condition is true, (i) $[d(u), u]_{1,\alpha} = 0$. (ii) $d(u)u = \alpha(u)d(u)$ (iii) $d(u^2) = \pm\, \alpha(u^2)$ (iv) $d(u^2) = 2d(u)\alpha(u)$, for all $u \in L$.

Preliminaries: Throughout the paper, N will be a 2 - torsion free σ- prime near-ring and 1, $\sigma \in$ Aut N. Let d be a nonzero (1,α)-derivation of N which commutes with σ and L be a nonzero σ- Lie ideal and a subring of N. Also, we will make some extensive use of the basic commutator identities.

$$[x, yz] = y\,[x, z] + [x, y]\,z$$
$$[xy, z] = [x, z]\,y + x[y, z]$$

$[xy, z]_{1,\alpha} = x[y, z]_{1,\alpha} + [x, \alpha(z)]y = x \quad [y, z] + [x, z]_{1,\alpha}\, y$

$[x, yz]_{1,\alpha} = \alpha(y)\,[x,z] + [x, y]_{1,\alpha}\,(z)$

$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$
$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$
$(x \circ (yz))_{1,\alpha} = (x \circ y_{1,\alpha} z - \alpha(y)\ [x, z]_{1,\alpha} = \alpha(y)(x \circ z)_{1,\alpha} + [x, y]_{1,\alpha} z$
$((xy) \circ z)_{1,\alpha} = x(y \circ z)_{1,\alpha} - [x, \alpha(z)]\ y = (x \circ z)_{1,\alpha}\ \alpha(y) + x[y, z]$

Lemma 2.1 : *If N is a 2-torsion free near-ring and L a nonzero lie ideal of N, then $2[N, N]L \subseteq L$ and $2L[N, N] \subseteq L$.*

Proof : Let $x, y \in N$ and $l \in L$. Then $l \circ [x, y] - [[l, x], y] + [[l, y], x] \in L$. and hence $2xyl - 2yxl \in L$ and we get $2[x, y]l \in L$, for all $x, y \in N$ and $l \in L$, that is $2[N, N]L \subseteq L$. Similarly, it is easy to see that $2l[x, y] = l[x, y] - [[l, y], x] + [[l, x], y] \in L$, for all $x, y \in N$ and $l \in L$, for all $x, y \in N$ and $u \in L$, and hence $2L[N, N] \subseteq L$.

Lemma 2.2: *Let N be a 2-torsion free σ- prime near-ring and L a nonzero σ-Lie ideal of N. If $aLb = \sigma(a)Lb = 0$ then $a = 0$ or $b = 0$.*

Proof : Assume that $a \neq 0$. Since $2[N, N]L \subseteq L$ by Lemma 2.1, then $2a[p, q]lb = 0$ for all $p, q \in N$, $l \in L$. This implies that $a[p, q]lb = 0$, (2.1)
for all $p, q \in N$, $l \in L$. Replacing q by qa in (2.1) because of $alb = 0$, we find that $aqaplb = 0$ and thus $aNaplb = 0$ (2.2) for all $p \in N$, $l \in L$.
On the other hand, from $\sigma(a)Lb = 0$, it follows that $\sigma(a)[p, qa]lb = 0$ which leads to $\sigma(a)qaplb = 0$ for all $p, q \in N$ and therefore $\sigma(a)Naplb = 0$, (2.3)
for all $p \in N$, $l \in L$. From the equations (2.2) and (2.3), because of $a \neq 0$, the σ- primeness of N yields $aplb = 0$ for all $p \in N$, $l \in L$. Accordingly
$aNlb = 0$, (2.4)
for all $l \in L$. Writing $q\sigma(a)$ instead of q in (1), we obtain $a[p, q\sigma(a)]lb = 0$ because of $\sigma(a)lb = 0$, we get $aq\sigma(a)plb = 0$ so that $aN\sigma(a)plb = 0$, (2.5)
for all $l \in L$. In view of $\sigma(a)lb = 0$, we find that $\sigma(a)[p, q\sigma(a)]lb = 0$ and thus $\sigma(a)q\sigma(a)plb = 0$ for all $p, q \in N$, $l \in L$. Hence $\sigma(a)N\sigma(a)plb = 0$, (2.6)
for all $p \in N$, $l \in L$. Using (2.5) and (2.6), because of $a \neq 0$, the σ- primeness of N yields $\sigma(a)plb = 0$ and therefore $\sigma(a)Nlb = 0$, (2.7)
for all $l \in L$. Again, because of equations (2.4) and (2.7), σ- primeness of N assures that $lb = 0$ for all $l \in L$. Hence it follows that $Lb = 0$. (2.8)
From $[l, p]b = 0$, by view of (2.8) we get $lpb = 0$ for all $p \in N$, $l \in L$ and thus
$lNb=0$, (2.9) for all $l \in L$ Since L is invariant under σ, from (2.9) it follows that $\sigma(l)Nb = 0$, (2.10)
for all $l \in L$. Using the σ- primeness of N, because of $l \neq 0$, equations (2.9) and (2.10) assure that $b = 0$.

Lemma 2.3 : *Let N be a 2 – torsion free σ-prime near-ring and L a nonzero σ- Lie ideal of N. If $[L, L] = 0$, then $L \subseteq Z(N)$.*

Proof : From $[2x[p, q], y] = 0$ it follows that $[x[p, q], y] = 0$ and thus $[x[p, q], y] = [x, y][p, q] + x[[p, q], y] = 0$, for all $p, q \in N$, $x, y \in L$. Hence.
$L[[p, q], y] = 0$, (2.11)
for all $p, q \in N$, $y \in L$. Since equation (2.11) is analogous to equation (2.8), arguing as in the Proof of Lemma 2.2, we arrive at
$[[p, q], y] = 0$, (2.12)
for all $l \in L$. Replacing q by qp in (2.12) we obtain $[[p, qp], y] = 0$ $[p, q][p, y] = 0$, (2.13)
for all $p, q \in N$, $y \in L$. Writing xq instead of q in (2.13) where $x \in L$, we obtain $[p, qx][p, y] = 0$ and thus $[p, x]q[p, y] = 0$ and we get
$[p, x]N[p, y] = 0$, (2.14)
for all $x, y \in L$, $p \in N$. Since $\sigma(L) = L$, replacing y by $\sigma(y)$ in (2.14), we get
$[p, x]N[p, \sigma(y)] = 0$, (2.15) for all $x, y \in L$, $p \in N$.
Let $p \in Sa_{\sigma}(N)$. From (15) it follows that
$[p, x]N\sigma[p, y] = 0$, (2.16)
for all $x, y \in L$. Using (2.14) together with (2.16), the σ - primness of N forces $[p, x] = 0$ for all $x \in L$. Accordingly
$[p, x] = 0$, (2.17)
for all $l \in S\,a_{\sigma}(N)$, $x \in L$. Let $p \in N$, since $p - \sigma(p) \in Sa_{\sigma}(N)$. (2.17) yields $[p - \sigma(p), p] = 0$, for all $x \in L$ and therefore
$[p, x] = [\sigma(p), x]$, (2.18)
for all $p \in N$, $x \in L$. Substituting $\sigma(p)$ for p in (2.15) and using (2.18) we get $[\sigma(p), x]N[\sigma(p), \sigma(y)] = 0$ for all $x, y \in L$, $p \in N$, which leads to
$[p, x]N\sigma[x, y] = 0$, (2.19)
for all $x, y \in L$, $p \in N$. Using the σ- primeness of N, equations (2.14) and (2.19) assure that $[p, x] = 0$ for all $p \in N$, $x \in L$ proving that $L \subseteq Z(N)$.

Lemma 2.4.: *Let N be a 2 – torsion free σ- prime near-ring and L a nonzero σ - Lie ideal of N. If $L \subseteq Z(N)$, then N is commutative*.

Proof : From $L \subseteq Z(N)$, it follows that $pl = lp$ implies $[p, l] \in L$ for all $p \in N$ and $l \in L$ and thus
$pql = p(ql) = q(pl)$. (2.20) Hence equation (2.20) yields $[p, q]l = 0$, (2.21)
for all $p, q \in N$ and $l \in L$. Replacing q by qt in (2.21) where $t \in N$, we get $[p, qt]l = 0$ and hence we get $[p, q]tl = 0$; there by $[p, q]Nl = 0$, (2.22)
for all $p, q \in N$ and $l \in L$. Since L is a σ- ideal, then (2.22) implies that $[p, q]N\sigma(l) = 0$, (2.23)
for all $p, q \in N$ and $l \in L$. In view of Lemma 2.2, because of $0 \neq L$, equation (2.22) together with equation (2.23) force $[p, q] = 0$ for all $p, q \in N$. Accordingly, N is commutative.

Lemma 2.5: *Let N be a 2 - torsion free σ- prime near-ring, L a nonzero σ-Lie ideal of N and d a nonzero $(1,\alpha)$-derivation of N. If d commutes with σ and $d(L) = 0$, then N is commutative.*

Proof : By the hypothesis, we obtain th

$d(up - pu) = 0$, for all $u \in L$, $p \in N$.
Expanding the term and using the hypothesis, we get
$(d(p), u)_{1,\alpha} = 0$, (2.24)
for all $x \in L$, $p \in N$. Replacing p by $2pv$, $v \in L$ in (2.24) and using (2.24), $d(L) = 0$, we have $0 = d(p)\alpha([v, u])$, for all $u, v \in L$, $p \in N$. Substituting pq, $q \in N$, for p in this equation and using this, we find that $0 = d(p)\alpha(q)\alpha([v, u])$ and so
$d(p)N\alpha([v, u]) = 0$, (2.25)
for all $u, v \in L$, $p \in N$. Writing $\sigma(p)$ by p in the last equation, we get $d(\sigma(p)N\alpha([v, u]) = 0$, for all $u, v \in L$, $p \in N$. Since d commutes with σ, the last equation follows
$d(\sigma(p)N\alpha([v, u]) = 0$, (2.26)
for all $u, v \in L$, $p \in N$. Applying the σ - primeness of N because of equation (2.25) and equation (2.26), we conclude that $d(p) = 0$ or $([v, u]) = 0$, for all $u, v \in L$, $p \in N$. Since d is a nonzero $(1, \alpha)$-derivation of N. We arrive at $[L, L] = 0$, and so N is commutative by Lemmas 2.3 and 2.4.

Results:

Theorem 3.1. *Let N be a 2-torsion free σ - prime near-ring and L a nonzero σ - Lie ideal of N. If N admits a nonzero $(1, \alpha)$-derivation d which commutes with σ such that $[d(u), u]_{1, \alpha} = 0$ for all $u \in L$* and $\alpha(L) = L$, *then N is commutative.*

Proof . Suppose that
$[d(u), u]_{1, \alpha} = 0$, (3.1) for all $u \in L$. Linearizing (3.1) and using this we obtain that
$[d(u), v]_{1, \alpha} + [d(v), u]_{1, \alpha} = 0$, (3.2) for all $u, v \in L$. Replacing u by uv in (3.2), we get $d(u)[v, \alpha(v)] + [d(u), v]_{1,\alpha,}v + \alpha(u)[d(v), v]_{1,\alpha} + [\alpha(u), \alpha(u)]_{1,\alpha}v + u[d(v), \alpha(v)]_{1,\alpha,} + [d(v), u]_{1,\alpha}v = 0$, for all $u, v \in L$.
Now combining (3.1) and (3.2) in the last equation, we find that
$0 = \alpha(u)[d(v), v]_{1,\alpha,} + [\alpha(u), \alpha(v)]d(v)$, (3.3)
for all $u, v \in L$. Again replacing v by vw in (3.3) and use (3.3), to get $0 = [\alpha(u), \alpha(v)]\, \alpha(w)d(v)$, for all $u, v, w \in L$. Since α is an automorphism of N, we see that $\alpha[u, v]L\alpha^{-1}(d(v)) = 0$, for all $u, v \in L$. Since L is nonzero σ - Lie ideal of N yields that $\sigma[u, v]L\alpha^{-1}(d(v)) = 0$, for all $u \in L$, $v \in L \cap Sa_\sigma(N)$. Hence we get
$[u, v]\, L\alpha^{-1}d(v) = \sigma[u, v]\, L\alpha^{-1}d(v) = 0$, (3.4)
for all $u \in L$, $v \in L \cap Sa_\sigma(N)$. By Lemma 2.2, we get either $[u, v] = 0$ for all $u \in L$ or $d(v) = 0$ for each $v \in L \cap Sa_\sigma(N)$. Let $v \in L$, as $v + \sigma(v), v - \sigma(v) \in L \cap Sa_\sigma(N)$ and $[u, v \pm \sigma(v)] = 0$, for all $u \in L$ or $d(v \pm \sigma(v)] = 0$. Hence we have $[u, v] = 0$ or $d(v) = 0$ for all $u, v \in L$. we obtain that L is union of two additive subgroups of U such that $J = \{v \in L / d(v) = 0\}$ and $K = \{v \in L / [u, v] = 0$ for all $u \in L\}$. Moreover, L is the set theoretic union of J and K. But a group cannot be the set theoretic union of two proper subgroups, hence $J = L$ or $K = L$. In the former case, we get N is commutative Lemma 2.5. In the latter case, $[L, L] = (0)$. That is $L \subseteq Z$ by Lemma 2.3 and N is commutative by Lemma 2.4.This completes the proof.

Theorem 3.2. *Let N be a 2-torsion free σ -prime near-ring and L a nonzero σ - Lie ideal of N. If N admits a nonzero (1,α)-derivation d which commutes with σ such that $d(u)u = \alpha(u)d(u)$, for all $u \in L$* and $\alpha(L) = L$, *then N is commutative.*

Proof. We have
$d(u)u = \alpha(u)d(u)$, (3.5)
for all $u \in L$. Replacing u by $u + v$ in (3.5) and using this, we get
$d(u)v + d(v)u = \alpha(u)d(v) + \alpha(v)d(u)$, (3.6)
for all $u, v \in L$. Writing uv for u in (3.6) and using (3.6), we obtain that $2\alpha(u)d(v)v = \alpha(u \circ v)d(v)$, for all $u, v, w \in L$. Taking wu instead of u in the above equation and using this, we have $\alpha\,[w, v]\alpha(u)d(v) = 0$, for all $u, v, w \in L$. Hence we arrive at $[w, v]L\alpha^{-1}(d(v)) = 0$, for all $u, v, w \in L$. Since L is nonzero σ - Lie ideal of N yields that $\sigma[w, v]L\alpha^{-1}(d(v)) = 0$, for all $u, v, w \in L$. Therefore, we get $[w, v]L\alpha^{-1}(d(v)) = \sigma[w, v]L\alpha^{-1}(d(v)) = 0$, for all $u \in L$, $v \in L \cap Sa_\sigma(N)$. The similar argument as used after equation (3.4), we get the required result.

Theorem 3.3. *Let N be a 2 - torsion free σ - prime near-ring and L a nonzero σ - Lie ideal of N. If N admits a nonzero (1, α)-derivation d which commutes with σ such that $d(u^2) = \pm\,\alpha(u^2)$, for all $u \in L$* and $\alpha(L) = L$, *then N is commutative.*

Proof **.** Linearizing the hypothesis , we get
$d(u)v + \alpha(u)d(v) + d(v)u + \alpha(v)d(u) = \alpha(uv + vu)$, (3.7)
for all $u, v \in L$. Replacing u by uv, for all $u \in L$ in (3.7) and applying this equation, we arrive at
$\alpha(u \circ v)d(u) = 0$, (3.8)
for all $u, v \in L$. Writing uw for u in (3.8) and using (3.8) we obtain that $\alpha([u, v])\alpha(w)d(v) = 0$, for all $u, v, w \in L$ and so $[u, v]L\alpha^{-1}(d(v)) = 0$, for all $u, v \in L$. Since L is a nonzero σ - Lie ideal of N yields that $\sigma[u, v]L\alpha^{-1}(d(v)) = 0$, for all $u \in L \cap Sa_\sigma(N)$. Therefore, we get $[u, v]L\alpha^{-1}(d(v)) = \sigma[u, v]L\alpha^{1}(d(v)) = 0$, for all $u \in L \cap Sa_\sigma(N)$. The similar arguments as used after equation (3.4), we get the required result.

Theorem 3.4. *Let N be a 2-torsion free σ - prime near-ring and L a nonzero σ - Lie ideal of N. If N admits a nonzero (1,α)-derivation d which commutes with σ such that $d(u^2) = 2d(u)\alpha(u)$, for all $u \in L$* and *$\alpha(L) = L$, then N is commutative.*

Proof **.** We get $d(u^2) = 2d(u)\alpha(u)$, for all $u \in L$. That is
$d(u)\,u + \alpha(u)d(u) = 2d(u)\alpha(u)$, (3.9)
for all $u \in L$. Linearizing (3.9) and using this we obtain
$d(u)v + \alpha(u)d(v) + d(v)u + \alpha(v)d(u) = 2d(u)\alpha(v) + 2d(v)\alpha(u)$, (3.10)

for all $u, v \in L$. Taking uv instead of u in (3.10) are using this equation, we have

for all $u, v \in L$. Letting u by uw in (3.11) and using (3.11), we arrive at

$\alpha([u, v])\alpha(w)d(v) = 0$, (3.12)

for all $u, v, w \in L$. That is $[u, v]L\alpha^{-1}(d(v)) = 0$, for all $u, v \in L$. Since L is nonzero σ - Lie ideal of N yields that $\sigma[u, v]L\alpha^{-1}(d(v)) = 0$, for all $v \in L$, $u \in L \cap S\, a_{?}$ (N).Therefore, we get $[u, v]L\alpha^{-1}(d(v)) = \sigma[u, v]L\alpha^{-1}(d(v)) = 0$, for all $v \in L$, $u \in L \cap S\, a_{\sigma}$ (N). Further application of similar arguments as used after (3.4) we get the required result.

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