# ON 2-$\mathcal{F}$-PRIMAL IDEALS IN NEAR RINGS

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***Abstract: In 1973, G. Shin considered the relation between rings in which the set of all nilpotent elements coincides with the prime radical. In 1990, Birkenmeir, Heatherly and Lee formulated the 2-primal condition in the context of left near-ring independently and were unaware of Shin's result for rings. In this paper, we introduce the concept of 2-f-primal ideal in near rings corresponding to the f -prime ideal in near rings and generalise different result for the same. Also, it can be seen that if f (0) =< 0 >, then 2-f -primal ideals are 2-primal ideals, but converse need not be true.***

***Keywords: f -nilpotent element, f -prime radical, 2-f –primal near ring, 2-f -primal ideal.***

## 1. INTRODUCTION

Throughout this article, $N$ denotes a zero symmetric near ring. For any subset $X$ of $N$, $<X>$ denotes the smallest ideal containing $X$. For preliminary definitions and results related to near rings, we refer Pilz [6]. Let $N$ be a near ring. Then a normal subgroup $I$ of $(N, +)$ is said to be (i) a left ideal if $a(b + i) - ab \in I$, for all $a, b \in N$, and $i \in I$; (ii) a right ideal if $ia \in I$, for all $a \in N$, and $i \in I$; (iii) an ideal if it is both left and right ideal of $N$. $N$ is said to be a zero-symmetric near ring if $a.0 = 0$, for all $a \in N$, where 0 is the additive identity in $N$. A subset $H$ of $N$ is said to be an $m$–system if, for every $h_1, h_2 \in H$, there exist $h_1^1 \in < h_1 >$ and $h_2^1 \in < h_2 >$ such that $h_1^1 h_2^1 \in H$. A subset $H$ of $N$ is said to be $f$–system if $H$ contains an $m$–system $H^*$, called kernel of $H$, such that for every $h \in H$, $f(h) \cap H^* \neq \phi$. An ideal $A$ of $N$ is said to be $f$–prime if $N \setminus A$ is an $m$–system. An ideal

$f(a)$ is a uniquely determined ideal by an element $a \in N$ which satisfies the following-:
(i) $a \in f(a)$
(ii) $x \in f(a) + A \Rightarrow f(x) \subseteq f(a) + A$, for any ideal $A$ of $N$.

Let $I$ be an ideal of near ring $N$ and consider the quotient near ring $N/I$. For a given ideal mapping $f$ on $N/I$, is a uniquely determined ideal defined by $f(x + I) = (f(x) + I)/I$, $x \in N$, which satisfies the two conditions mentioned in the definition of ideal $f(a)$.

An ideal $I$ of a near ring $N$ is called 2-primal ideal if $P(N/I) = N(N/I)$, where $P(N/I)$ and $N(N/I)$ are prime radical and the set of all nilpotent elements of near ring $N/I$ respectively. $N$ is called 2-primal if the zero ideal of $N$ is 2-primal.

**Lemma 1.1** Let $I$ be an ideal of a near ring $N$. Also, let $h: N \rightarrow N/I$ be a canonical epimorphism and let $P$ be an another ideal of the near ring $N$ containing $I = \ker(h)$. Then the following are equivalent:
(a) $P$ is an $f$-prime ideal in the near ring $N$.

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(b) *h(P )*is an *f*-prime ideal in the near ring *N /I*.

**Proof**. We show (b) ⇒ (a). Let $h(P) = \{ h(p) \mid p \in P \} = P/I$ be an *f*-prime ideal in *N* /I. We show that *P* is an *f*-prime ideal in the near ring *N*. Since *P/I* is an *f*-prime ideal in *N/I*. So, $(N/I) \setminus (P/I) = \{ r + I \mid r \notin P \}$ (say $= S$). Now, *S* is an *f*–system as there exists an *m*–system $S^*$ ( *S*) such that

$f(s) \cap S^* \neq \phi$, for all $s \in S$. Clearly, *S* does not contain any element of *P*. We show that *P* is an *f*–prime ideal of *N*. If possible, let *P* is not *f*-prime ideal of *N*, then $N \setminus P$ is not an *f*–system. So, for any *m*–system $S^* \subseteq N \setminus P$, we have $f(s) \cap S^* = \phi$, for $s \in S$. Now, we consider $S` = \{ r \mid r \in S \}$. Then $S`$ is an *f* – system with kernel $S`$ itself. Also, $S` \subseteq N \setminus P$. But $f(s) \cap S` \neq \phi$, for all $s \in S$, which is a contradiction.

Now, we show (a) ⇒ (b). Let *P* be an *f*–prime ideal of the near ring *N*. Then $P` = N \setminus P$ is an *f*-system with kernel itself. Also $P` = \{x \in N \mid x \notin P\}$. Consider $P`` = \{ x \in P` \mid h(x)\} = \{x \in P` \mid x + I \}$. Then $P``$ is an *m*–system as well as an *f*–system. Also, $P`` = \{x \in P` \mid x + I \} = \{ x \in N, x \notin P \mid x + I \} = (N/I) \setminus (P/I)$. So, *P/I* is an *f*-prime ideal in *N /I*.

Hence the result.
Since, all ideals are kernels of homomorphisms in near rings, so, we have the following result.

**Theorem 1.2** Let *I* be an ideal of the near ring *N*. Also, let $h: N \rightarrow N/I$ be a canonical epimorphism. Then there is one-one correspondence between the set of all *f*-prime ideals of *N*, containing *I* and the set of all *f*-prime ideals of *N/I*.

**Proof.** Let $g: N \rightarrow N/I$ be the natural epimorphism defined by $g(x) = x + I, x \in N$. Now, if *P* be any *f*–prime ideal of *N* containing *I*, then *h(P)* is an *f*–prime ideal of *N/I* [ using Lemma (1.1) ]. Also, $h(P) = \{ h(p) \mid p \in P\} = \{ p+I \mid p \in P \} = P/I$. Let *K* be the set of all *f*-prime ideals of *N* which contains *I* and *K`* be the set of all *f* -prime ideals of *N /I*.

Let $\Phi : K \rightarrow K`$ such that $\Phi(P) = h(P) = P/I$.

Then $\Phi$ is well defined and one-one.

Let $P` \in K`$ be an *f*-prime ideal of *N /I*. Let $A = \{a \in P \mid h(a) \in P` \}$. We claim *A* is a required pre-image of *P`* under $\Phi$. It is clear that *A* is an *f*–prime ideal of *N /I* [using Lemma 1.1]. Also, $I \subseteq A$. Let $a \in I$. Then $h(a) = a + I = I$ (0 of *N /I*) $\in P`$. So, $a \in A$, this implies that *A* is a member of *K*.

Hence the result.

**Theorem 1.3** Let *I* be an *f*-prime ideal of *N* . If $f(a).f(b) \subseteq I \Rightarrow a \in I$ or $b \in I$.

**Proof**. See the Lemma (1.2) of [4]

**Example 1.4** The union of two *f*-prime ideal need not be an *f* -prime ideal in a near ring *N*. Let $A =< 3 >$ and $B = < 4 \cup 6 >$. Take $f(s) = <s \cup 3>$, for $s \in N$. Then *A* and *B* are *f*-prime ideals, with kernel $A^* = N \setminus A$ and $B^* = < 3 > - <6 >$ respectively, which are *m*-systems. Also, then $f(2).f(2) \subset P \cup Q = < 3 \cup 4 >$. But $2 \notin P \cup Q$, which contradicts the Theorem (1.3).

## 2. 2-*F*-PRIMAL NEAR RING

Bhavanari [1, 2] has defined the following:

**Definition 2.1** Let *M* be a *Γ*-near ring. An element $a \in M$ is said to be *f* –nilpotent if *f*(*a*) is nilpotent.

**Definition 2.2** A subset *H* of *M* is said to be nilpotent if $H^n = \{0\}$ i.e. $H\Gamma H\Gamma H..n$times...$\Gamma H = \{0\}$, for some $n \geq 2$.

By using above definitions for near ring *N*, considering $\Gamma = (.)$, we conclude for the ideal $f(a)$ , being a subset of $N$, $f(a)$ is nilpotent means $[f(a)]^n = \{0\}$, for some $n \geq 2$. i.e. $f(a).f(a).f(a)$..n times. $f(a) = \{0\}$.

**Definition 2.3** The *f* -prime radical of near ring *N* is defined as the intersection of all *f* –prime ideal of *N* and is denoted by *f*–rad(*N*).

**Note 2.4** {Reddy and Bhavanari [2]} For an ideal *I* of a near ring *N*, *f*-rad(*I*)= $\{x \in N \mid$ every *f*- system containing *x* must contains an element of *I*.}

Now, we prove the following results:

**Proposition {2.5}** The intersection of all *f*-prime ideals of *N* is contained in the set of all *f*-nilpotent element of *N* .

**Proof**. Let $a \notin$ the set of all *f*– nilpotent element of *N*.

$\Rightarrow f(a)$ is not nilpotent in *N*.

$\Rightarrow \{f(a)\}^n \neq (0)$, for any $n \in Z_+$. Let $I = f(a)$.Then $I^n \neq (0)$, for any $n \in Z_+$. Let $X = \{I^n \mid n \in Z_+\}$. Then, *X* is an *f*–system with Kernel $X = X^*$ and if we take, $A = (0)$. Then $(0) \cap X = \phi$. So, by using Lemma (1.5 (a)) of [2], *A* is contained in a maximal ideal *P*, which does not meet *K*. In this case, ideal *P* is an *f* –prime ideal. i.e. $(0) \subseteq P$, where $P \cap X = \phi$. Which shows that $a \notin P \Rightarrow a \notin \cap P$, where *P* is the *f*–prime ideal. So, $\cap P \subseteq$ the set of all *f*–nilpotent element of *N* .

**Example 2.6** The set of all *f*–nilpotent element of *N* need not contained in the intersection of all *f*–prime ideals of *N*.

Let $N = N_1 \oplus N_2 \oplus N_3$, where $N_1, N_2, N_3$ are near rings. Let *a* be an *f*- nilpotent element such that $a \in N_1$. Then *a* is also a nilpotent element. Let $S^* = \{a, a^2, a^3$

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........$a^n = 0\}$, for some $n \geq 2$. Then $S^*$ is an *m*-system and $S^* \subseteq N \setminus N_2$. Let $f(x) = \langle x \rangle$. Then $f(x) \cap S^* \neq \phi$, $\forall\ x \in N \setminus N_2$.

So, $N \setminus N_2$ is an *f*-system. $\Rightarrow N_2$ is an *f*-prime ideal. Thus, we have an *f* -nilpotent element, which is not in at least one of the *f*-prime ideal.

Now, based upon the above observation, we introduce the following definitions :

**Definition 2.7** If the intersection of all *f*–prime ideal of *N* is precisely equals the set of all *f* –nilpotent element of *N*. Then *N* is called a 2-*f*– primal near ring.

**Definition 2.8** A near ring which contains no non zero *f* –nilpotent element is called *f*-reduced near ring.

**Example 2.9** Every *f*–reduced near ring is a 2-*f*–primal near ring.

For this, we show set of all *f*– nilpotent element of $N \subseteq f$– rad($N$). Since *N* is an *f*–reduced near ring. So, 0 (zero element of *N)* is the only *f*–nilpotent element. We show that $0 \in f$–rad($N$). Let if, $0 \notin f$–rad($N$). Then, by using Note (2.4), $\exists$ an *f*–system which contains 0 but does not contains any element of *N*, a contradiction. Hence the result.

**Observation 2.10** All *f*-reduced near rings are reduced. Let *N* be reduced. Then there does not exist any non trivial *f*-nilpotent element. i.e. for any $0 \neq a \in N$, $(f(a))^n \neq \{0\}$, for any $n \geq 2 \Rightarrow a^n \in (f(a))^n \neq \{0\}$, for any $n \geq 2 \Rightarrow \exists$ no non-trivial nilpotent element. Hence *N* is reduced.

**Example 2.11** If *N* has no non zero divisor, then also, it is 2-*f* -primal near ring. Let $a \in$ the set of all *f*–nilpotent element of *N* . $\Rightarrow (f(a))^n = \{0\}$, for some $n \geq 2$.

$\Rightarrow a.a.a..n$ times...$a \in f(a).f(a)...n$times..$f(a) = (f(a))^n = \{0\}$.
$\Rightarrow a^n \in \{0\} \Rightarrow a = 0$. Also, as discussed before,
$0 \in f$–rad($N$). Hence the result.

**Definition 2.12** When the intersection of all *f*–prime ideal of *N/I* is precisely equals the set of all *f*-nilpotent element of *N/I*. We call *I* is a 2-*f*–primal near ring. And using Lemma (1.1), we have the following:

When the intersection of all *f*–prime ideal of *N* containing *I* is precisely equals the set of all *f*–nilpotent element of *N /I*. i.e. $f$-rad($I$)= $N_f$(N /$I$).

**Proposition 2.13** If $[f(a)]^n \subseteq I \Rightarrow f(a) \subseteq f$ -rad($I$). Then *I* is a 2-*f* –primal ideal of the near ring *N*.

**Proof.** Let $a + I \in N_f$ (N /$I$). $\Rightarrow a + I$ is an *f*-nilpotent element. $\Rightarrow$ $f(a + I)$is nilpotent. Then $[f(a+I)]^n = [(f(a)+I)/I]^n = I$, for some $n \geq 2$, which implies $[f(a)]^n \subseteq I$, for some $n \geq 2$. Now, using given condition, we have $f(a) \subseteq f$-rad($I$). Then $a \in f(a) \subseteq f$ *-rad(I)* $= \cap\ P$ , where intersection is taken over all *f* -prime ideals of *N* containing ideal $I$ of $N$. This implies, $a \in P$, where $P$ is an $f$ -prime ideal of $N$ containing $I$. $\Rightarrow a + I \subseteq P$. But $P$ was arbitrary, so $a + I \in f$-rad($I$). Hence, $I$ is a 2-$f$ -primal ideal of $N$.

**Proposition 2.14** Let $I$ be a 2-$f$ -primal ideal of the near ring $N$. If $[f(a)]^n \subseteq I$, then $f(a) \subseteq f$-rad($I$).

**Proof**. Let $[f(a)]^n \subseteq I$, which implies $[(f(a)+I)/I]^n$= I. And so,

$[f(a+I)]^n = I$. This implies $a+I \in N_f$ (N $/I$) = $f$–rad($I$). So, $a+I \in \cap P$, where intersection is taken over all $f$-prime ideals of $N$ containing ideal $I$ of $N \Rightarrow a \in P$. Now, if possible, let $f(a)$ is not contained in $f$-rad($I$). This implies that $f(a)$ is not contained in at least one $f$- prime ideal say $P$. So, $N \setminus P$ is an $f$-system, and $a \in f(a) \subseteq N/P$, which contradicts $a \in P$. Hence the result.

**Theorem 2.15** The near ring $N$ is 2-$f$- primal $\Leftrightarrow$ $N/f$– rad($N$) is $f$-reduced.

**Proof.** Let $N$ is 2-$f$ -primal. To show $N_f(N/f -$ rad(N)) = $f$– rad($N$). Let $x$+ $f$–rad($N$) $\in N_f(N/f$ –rad(N)) $\Rightarrow [f(x+ f\text{–rad}(N))]^n = f$–rad($N$). Then, we have $[f(x)]^n \subseteq f$– rad($f$ – rad($N$)). Now, being an $f$-radical of a zero ideal, $f$–rad($N$) is an ideal and using the Proposition (2.14), we have $f(x) \subseteq f$– rad($f$– rad($N$) = $f$– rad($N$) $\Rightarrow x \in f$– rad($N$). Also, $f$– rad($N$) is the zero element of $N/f$ – rad($N$). So, $x$ + $f$-rad($N$) $\subseteq N_f(N/f$– rad($N$). Hence the result.

Conversely, Let ($N/f$– rad($N$)) is $f$-reduced, so, $N_f(N/f$– rad($N$) = $f$–rad($N$). To show $N_f(N) \subseteq f$– rad($N$).

Let $x \in N_f(N) \Rightarrow x + f$– rad($N$) is an $f$- nilpotent element in $N/ f$-rad($N$) $\Rightarrow$ $x$+$f$–rad($N$) $\in N_f(N/f$–rad($N$) = $f$–rad($N$) $\Rightarrow x \in f$– rad($N$)). Hence proved.

**Proposition 2.16** The intersection of two 2-$f$ -primal ideals of near ring $N$ is again a 2-primal ideal of $N$, provided every $f$-system in $N$ has a densed kernel**.**

**Proof.** Let $I_1, I_2$ be two 2-primal ideals of $N$.

Let $x + (I_1 \cap I_2) \in N_f(N/(I_1 \cap I_2)) \Rightarrow [f(x)]^n \subseteq (I_1 \cap I_2)$, by using previous argument. Then, this implies $[f(x)]^n \subseteq I_1$ and

$[f(x)]^n \subseteq I_2$. Now, using Proposition (2.14), $f(x) \subseteq f$ –rad($I_1$) and $f(x) \subseteq f$–rad($I_2$) $\Rightarrow f(x) \subseteq f$–rad($I_1$)$\cap f$–rad($I_2$). But by using Theorem (3.1) of [4], $f(x) \subseteq f$–rad($I_1 \cap I_2$ ) $\Rightarrow x \in f$–rad($I_1 \cap I_2$). Thus, $x$+ ($I_1 \cap I_2$)$\in f$–rad($I_1 \cap I_2$ ). Hence proved.

**Remark 2.17** If $f(a)$ is nilpotent then $a$ is nilpotent. But converse need not be true. For this, we have the following Example:

**Example 2.18** Let $N$ = {0, $a$, $b$, $c$} be a Klein's four group with addition and multiplication defined as follows:

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| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | b | c |
| b | b | c | 0 | a |
| *c* | *c* | *b* | *a* | *0* |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | a |
| b | 0 | a | b | b |
| *c* | *0* | *a* | *b* | *c* |

Then ($N$ , +, .) is a near ring. (see Pilz [6], P. 408] scheme 1). Take $f(a) = <a>$, then $a$ is nilpotent but it is not $f$-nilpotent.

Also, this example leads that a 2-primal ideal need not be a 2-$f$ - primal ideal.

**Theorem 2.19** If $f(0) =< 0 >$, then every 2-$f$ -primal ideal $I$ of a near ring $N$ is a 2-primal ideal.

Proof. Let $f(0) =< 0 >$. We show $N(N/I) \subseteq P(N/I)$. But $\{ x \mid x$ is a nilpotent element$\} \subseteq \{ x \mid x$ is an $f$-nilpotent element $\} \subseteq \{$ Intersection of all $f$-prime ideals of containing $I \} \subseteq$

{Intersection of all prime ideals of containing $I$} {usingTheorem 2.3(III), Reddy and Bhavanari [2] }. Hence the result.

But the converse of above need not be true. For this, we have the following Example:

Example 2.20 Let $N = \{0, a, b, c\}$ be a Klein's four group with addition and multiplication defined as in Example (2.18). Then <0 >is 2-$f$ -primal, where $f(a) = <a, c>$. Clearly, $f(0) \neq <0>$. Since there does not exist any $f$–nilpotent element, so, $\phi \subseteq$ the set of all $f$-prime ideals of $N$. But

<0 >is not 2-primal, as the set of nilpotent element is $\{0, a\}$. But the intersection of all prime ideal will be 0, since (0) is the smallest prime ideal. And $\{0, a\}$ is not contained in $\{0\}$.

**Theorem 2.21** Let $I$ be an ideal of near ring $N$ such that $I \subseteq f$-rad($N$ ). Then $I$ is an 2-$f$ -primal ideal $\Leftrightarrow$ $N$ is an 2-$f$- primal ring.

**Proof.** Let $N$ be an 2-$f$ -primal ring and $I$ be an ideal of $N$ such that $I \subseteq f$-rad($N$ ). We show $N_f(N/I) = f$– rad($I$). Let $x + I \in N_f(N/I) \Rightarrow [f(x + I)]^n = I$, for some $n \geq 2$, which implies

$[f(x)]^n \subseteq I \subseteq f$–rad($N$) $= \cap P$ , where intersection is taken over all $f$-prime ideals of $N$.

$\Rightarrow [f(x + I)]^n = P \Rightarrow x \in P$, by using the Theorem 2.15. Also, $I \subseteq N$. So, $x + I \in P/I$ $\Rightarrow x + I \in \cap P/I$, where intersection is taken over all $f$-prime ideals of $N$ containing $I$

$\Rightarrow x + I \in f$ –rad($I$). The converse part is obvious. Hence the result.

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