THREE POLYNOMIALS IN ONE MATRIX

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Abstract : The Wiener Index of a graph is defined as the sum of the smallest distance d(x, y) between all pair of vertices in a connected graph and a q-analogue of this index is termed as the Wiener Polynomial. Similar to Wiener Index, the Detour Index of a graph is defined as the sum of the longest distance D(x, y) between all pairs of vertices in a connected graph, and a q-analogue of this Detour index is termed as the Detour Polynomial. The circular distance $d^0(x, y)$ between two vertices x, y of a graph is defined as D(x, y) + d(x, y). The circular index of a graph is defined as the sum of circular distances $d^0(x, y)$ between all pairs of vertices in a connected graph, a q-analogue of this circular Index is termed as circular polynomial.

Keywords : Circular Index, Circular Polynomial, Detour Polynomial, Detour Index, Wiener Index, Wiener-Detour Matrix and Wiener Polynomial.

1. INTRODUCTION

The circular distance $d^0(x, y)$ between two vertices x, y of a graph is defined as D(x, y) + d(x, y), where D(x, y) is the detour index and d(x, y) is the Wiener Index. This circular distance plays an important role in logistical management. The transportation of commodities from a warehouse located at a town 'x' into a destination located at a town 'y' includes, a goods carrier which takes a long trip to distribute the goods from the town x to y covering all the towns between x and y. On the return trip, the shortest route may be selected to arrive to the warehouse x. The goods carriers trip can be represented by a graph 'G', where the vertices correspond to the towns and two vertices are adjacent in G if and only if there is a direct road connecting the corresponding towns which does not pass through any other town. The main aim of a goods carrier is to minimize the fuel and vehicle costs to distribute the goods. In this paper circular index of a graph is defined using Wiener Index and Detour Index. Further the circular index of certain graphs has been found using circular polynomial.

2. CIRCULAR INDEX OF SOME STANDARD GRAPHS

Definition 2.1 : Let D(u,v) denote the longest distance between two vertices u,v $\in$ V(G). The Detour Distance Polynomial of a graph G with q edges is denoted by $DP(G:q) = \sum_{u,v \in G} q^{D(u,v)}$. The Detour index is given by DI = $\Sigma DP'(G:1)$ where $'$ denotes the derivative of DP(G:q) with respect to q.

Definition 2.2 : Let d(u,v) denote the shortest distance between two vertices u,v $\in$ V(G). The Wiener polynomial of a graph G with q edges is denoted by WP(G:q)

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and is defined as $WP(G:q) = \sum_{u,v \in V(G)} q^{d(u,v)}$. The Wiener index of G is WI = ΣWP′(G:1) where ′ denotes the derivative of WP(G:q) with respect to q.

Definition 2.3 : Let $d^0(x, y)$ denote the circular distance between two vertices u, v ∈ V(G). The circular polynomial of a graph G with q edges is denoted by CP(G:q) and is defined as CP(G:q) = D(u,v) + d(u,v). The circular index of G is CI = CP′(G:1), where ′ denotes the derivative of CP(G:q) with respect to q.

Definition 2.4 : The Detour Matrix (DM) is a lower (or) upper triangular matrix whose elements are Detour Distances D(u,v).

Definition 2.5 : The Wiener Matrix (WM) is a lower (or) upper triangular matrix whose elements are Wiener Distance d(u,v).

Definition 2.6 : A Wiener – Detour Matrix of G is a square matrix of order 'p' with entries zero along the principle diagonal above the principle diagonal the entries are $D(u_i,v_j)$ and below the principle diagonal the entries are $d(u_i,v_j)$ where i ≠ j.

Definition 2.7 : A circular matrix of G is a lower or upper triangular matrix whose elements are $d^0(x, y)$.

Theorem 2.8 : The Detour Polynomial Wiener polynomial and circular polynomial of cycle graph C_n are respectively.

(i) $$DP(C_n:q) = \begin{cases} nq^{n-1} + nq^{n-2} + \dots + nq^{\frac{n+3}{2}} + nq^{\frac{n+1}{2}}, & \text{if n is odd} \\ nq^{n-1} + nq^{n-2} + \dots + nq^{\frac{n+2}{2}} + \frac{n}{2}q^{\frac{n}{2}}, & \text{if n is even} \end{cases}$$

(ii) $$WP(C_n:q) = \begin{cases} nq + nq^2 + nq^3 + \dots + nq^{\frac{n-3}{2}} + nq^{\frac{n-1}{2}}, & \text{if n is odd} \\ nq + nq^2 + \dots + nq^{\frac{n-2}{2}} + \frac{n}{2}q^{\frac{n}{2}}, & \text{if n is even} \end{cases}$$

(iii) $$CP(C_n:q) = \left(\frac{n^2 - n}{2}\right)q^n, \; n \geq 3.$$

Proof : Let $V(C_n) = \{u_i / 1 \le i \le n\}$ and

$E(C_n) = \{u_i u_{i+1} / 1 \le i \le n-1\} \cup \{u_n u_1\}$

be the vertex set and edge set of C_n respectively.

Case (i) : n is odd

The Wiener- Detour matrices of C_3, C_5, C_7 are respectively given in fig (i), fig (ii) and fig (iii).

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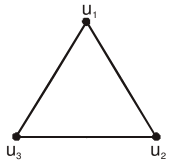


Fig.1 - C_3

WDM(C_3)

	u_1	u_2	u_3
u_1	0	2	2
u_2	1	0	2
u_3	1	1	0

CM(C_3)

	u_1	u_2	u_3
u_1	0	–	–
u_2	3	0	–
u_3	3	3	0

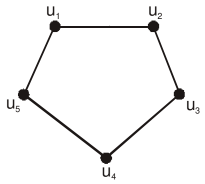


Fig.2 - C_5

WDM(C_5)

	u_1	u_2	u_3	u_4	u_5
u_1	0	4	3	3	4
u_2	1	0	4	3	3
u_3	2	1	0	4	3
u_4	2	2	1	0	4
u_5	1	2	2	1	0

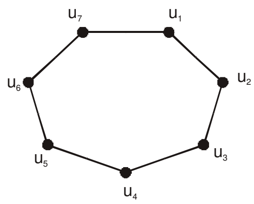


Fig.3 - C_7

WDM(C_7)

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	0	6	5	4	4	5	6
u_2	1	0	6	5	4	4	5
u_3	2	1	0	6	5	4	4
u_4	3	2	1	0	6	5	4
u_5	3	3	2	1	0	6	5
u_6	2	3	3	2	1	0	6
u_7	1	2	3	3	2	1	0

$DP(C_3:q) = 3q^2$; DI = 6 ;

$DP(C_5:q) = 5q^3 + 5q^4$; DI = 15 + 20 = 35

$WP(C_3:q) = 3q^2$; WI = 3 ;

$WP(C_5:q) = 5q + 5q^2$; WI = 5 + 10 = 15

$CP(C_3:q) = 3q^3$; CI = 9; $CP(C_5:q) = 10q^5$; CI = 50

$DP(C_7:q) = 7q^4 + 7q^5 + 7q^6$; DI = 28 + 35 + 42 = 105

$WP(C_7:q) = 7q + 7q^2 + 7q^3$; WI = 7 + 14 + 21 = 42

$CP(C_7:q) = 21q^7$; CI = 147

In general, when 'n' is odd

$$DP(C_n:q) = nq^{n-1} + nq^{n-2} + \ldots + nq^{\frac{n+3}{2}} + nq^{\frac{n+1}{2}}$$

$$WP(C_n:q) = nq + nq^2 + nq^3 + \ldots + nq^{\frac{n-3}{2}} + nq^{\frac{n-1}{2}}$$

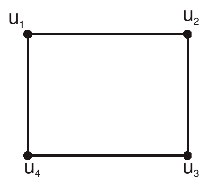
$$CP(C_n:q) = \left(\frac{n^2 - n}{2}\right) q^n, \; n \geq 3 \text{ \& n is odd.}$$

Hence Detour Index, Wiener Index and Circular Index are,

$$DI = \frac{3n^3 - 4n^2 + n}{8} \; ; \; WI = \frac{n(n^2-1)}{8} \; ; \; CI = \frac{n^2(n-1)}{2} ; \; n \geq 3 \text{ and n is odd.}$$

Case (i) : n is even

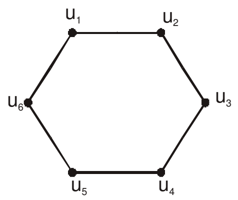
The Wiener - Detour matrices of C_4, C_6, C_8 are respectively given in fig (iv), fig (v) and fig (vi).



WDM(C_4)

	u_1	u_2	u_3	u_4
u_1	0	3	2	3
u_2	1	0	3	2
u_3	2	1	0	3
u_4	1	2	1	0

Fig.4 - C_4



WDM(C_6)

	u_1	u_1	u_1	u_1	u_1	u_1
u_1	0	5	4	3	4	5
u_2	1	0	5	4	3	4
u_3	2	1	0	5	4	3
u_4	3	2	1	0	5	4
u_5	2	3	2	1	0	5
u_6	1	2	3	2	1	0

Fig.5 - C_6

WDM(C_8)

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	0	7	6	5	4	5	6	7
u_2	1	0	7	6	5	4	5	6
u_3	2	1	0	7	6	5	4	5
u_4	3	2	1	0	7	6	5	4
u_5	4	3	2	1	0	7	6	5
u_6	3	4	3	2	1	0	7	6
u_7	2	3	4	3	2	1	0	7
u_8	1	2	3	4	3	2	1	0

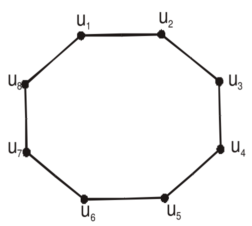


Fig.6 - C_8

$DP(C_4:q) = 2q^2 + 4q^3$; $DI = 4 + 12 = 16$

$WP(C_4:q) = 4q + 2q^2$; $WI = 4 + 4 = 8$

$CI(C_4:q) = 6q^4$; $CI = 24$

$DP(C_6:q) = 3q^3 + 6q^4 + 6q^5$; $DI = 9 + 24 + 30 = 63$

$WP(C_6:q) = 6q + 6q^2 + 3q^3$; $WI = 6 + 12 + 9 = 27$

$CP(C_6:q) = 15q^6$; $CI = 80$

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$DP(C_8:q) = 4q^4+8q^5+8q^6+8q^7$; DI = 16+40+48+56 = 160

$WP(C_8:q) = 8q+8q^2+8q^3+4q^4$; WI = 8+16+24+16 = 64

$CP(C_8:q) = 28q^8$; CI = 224

In general, when ‘n’ is even

$$DP(C_n:q) = nq^{n-1} + nq^{n-2} + \ldots + nq^{\frac{n+2}{2}} + \frac{n}{2}q^{\frac{n}{2}}$$

$$WP(C_n:q) = nq + nq^2 + \ldots + nq^{\frac{n-2}{2}} + \frac{n}{2}q^{\frac{n}{2}}$$

$$CP(C_n:q) = \left(\frac{n^2-n}{2}\right)q^n,\ n \geq 4 \ \&\ n \text{ is even.}$$

Hence Detour Index, Wiener Index and Circular Index are respectively.

$$DI = \frac{3n^3-4n^2}{8}; \quad WI = \left(\frac{n}{2}\right)^3; \quad CI = \frac{n^2(n-1)}{2}$$

Theorem 2.9 : For the Complete Graph K_n,

$$DP = \frac{n(n-1)}{2}q^{n-1},\ WP = \frac{n(n-1)}{2}q \text{ and } CP = \frac{n(n-1)}{2}q^n$$

Proof : Since G = K_n is the Complete Graph of ‘n’ vertices, the Detour Polynomial, Wiener polynomial and circular polynomial are $DP(K_n:q) = \frac{n(n-1)}{2}q^{n-1}$; $WP(K_n:q) = \frac{n(n-1)}{2}q$; $CP(K_n:q) = \frac{n(n-1)}{2}q^n$. The Detour Index, Wiener Index and Circular Index are the first derivative of the corresponding polynomial at q = 1. Hence

$$DI = \frac{n(n-1)^2}{2}; \quad WI = \frac{n(n-1)}{2}; \quad CI = \frac{n^2(n-1)}{2}$$

Theorem 2.10 : For any tree T, W(T:q) = D(T:q) and CI = 2WI = 2DI.

Proof : Since there is exactly one path between every pair of vertices, DI = WI ; ∴ CI = 2WI = 2DI.

Example 1 : For the Double Star Graph $K_{1,m,m}$

$WP(K_{1,m,m}:q) = DP(K_{1,m,m} : q)$

$$= \frac{m}{2}\left[4q+(m+1)q^2+(m-1)q^3+(m-1)q^4\right]$$

$$DI=WI=\frac{m}{2}\left[4+2(m+1)+3(m-1)+4(m-1)\right]=\frac{m}{2}(9m-1)\ ;\ CI=m(9m-1)$$

3. CIRCULAR INDEX OF PRODUCT GRAPH P_2 X C_N

Theorem 3.1 : The Detour Polynomial, the Wiener polynomial and the circular polynomial for Cartesian product of P_2 x C_n is

$$DP(P_2 \times C_n:q) = \begin{cases} n(2n-1)q^{2n-1}, & \text{if n is odd and } n \geq 3 \\ n(n-1)q^{2n-2} + n^2q^{2n-1}, & \text{if n is even and } n \geq 4. \end{cases}$$

$WP(P_2 \times C_n:q)$

$$= \begin{cases} 3nq+4n\left(q^2+q^3+...+q^{\frac{n-1}{2}}\right)+2nq^{\frac{n+1}{2}}, & \text{if n is odd and } n \geq 3 \\ 3nq+4n\left(q^2+q^3+...+q^{\frac{n-2}{2}}\right)+3nq^{\frac{n}{2}}+nq^{\frac{n+2}{2}}, & \text{if n is even and } n \geq 4 \end{cases}$$

$CP(P_2 \times C_n:q)$

$$= \begin{cases} 3nq^{2n}+4n\left(q^{2n+1}+q^{2n+2}+...+q^{\frac{5n-3}{2}}\right)+2nq^{\frac{5n-1}{2}}, & \text{if n is odd and } n \geq 3 \\ \left(\frac{3n^2+6n+24}{4}\right)q^{2n}+\left(\frac{3n^2-10n}{2}\right)q^{2n+2}, & \text{if n is even and } n \geq 4 \end{cases}$$

and DI = $DP'(P_2 \times C_n:1)$, WI = $WP'(P_2 \times C_n:1)$,
CI = $CP'(P_2 \times C_n:1)$.

Proof : Let G be the Cartesian product of the path graph P_2 and cycle graph C_n, $n \geq 3$. The vertex set and edge set of G = P_2 x C_n are given below,
$V(G) = \{w_1, w_2, w_3, \ldots, w_{2n}, n \geq 3\}$
$E(G) = \{w_iw_{i+1} ; 1 \leq i \leq n-1\} \cup \{w_iw_{i+1} ; n+1 \leq i \leq 2n-1\}$
$\cup \{w_1w_n, w_1w_{n+1}, w_{n+1}w_{2n}, w_nw_{2n}\}$

The vertex set of G has 2n vertices and the edge set has 3n edges.

Case (i) : n is odd and $n \geq 3$
The graph P_2 x C_3 is illustrated in fig (vii)

WDM(P_2 x C_3)

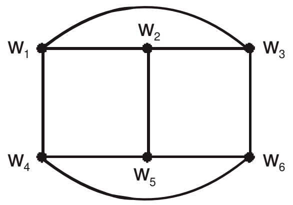

Fig.7 - P_2 x C_3

	w_1	w_2	w_3	w_4	w_5	w_6
w_1	0	5	5	5	5	5
w_2	1	0	5	5	5	5
w_3	1	1	0	5	5	5
w_4	1	2	2	0	5	5
w_5	2	1	2	1	0	5
w_6	2	2	1	1	1	0

$DP(P_2 \times C_3:q) = 15q^5$; DI = 75
$WP(P_2 \times C_3:q) = 9q + 6q^2$; WI = 21
$CP(P_2 \times C_3:q) = 9q^6 + 6q^7$; CI = 96

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The graph P_2 x C_5 is illustrated in fig (viii)

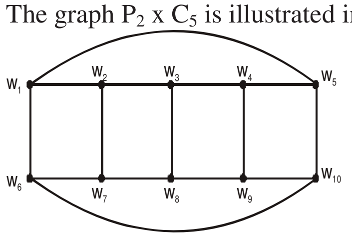


Fig.8 - P_2 x C_5

WDM(P_2 x C_5)

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
w_1	0	9	9	9	9	9	9	9	9	9
w_2	1	0	9	9	9	9	9	9	9	9
w_3	2	1	0	9	9	9	9	9	9	9
w_4	2	2	1	0	9	9	9	9	9	9
w_5	1	2	2	1	0	9	9	9	9	9
w_6	1	2	3	3	2	0	9	9	9	9
w_7	2	1	2	3	3	1	0	9	9	9
w_8	3	2	1	2	3	2	1	0	9	9
w_9	3	3	2	1	2	2	2	1	0	9
w_{10}	2	3	3	2	1	1	2	2	1	0

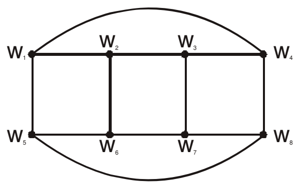
DP(P_2 x C_5:q) $= 45q^9$; DI = 405
WP(P_2 x C_5:q) $= 15q + 20q^2 + 10q^3$; WI = 85
CP(P_2 x C_5:q) $= 15q^{10} + 20q^{11} + 10q^{12}$; CI = 490
Therefore when n is odd and $n \geq 3$

$$DP(G:q) = n(2n-1)q^{2n-1}$$

$$WP(G:q) = 3nq+4n\left(q^2+q^3+...+q^{\frac{n-1}{2}}\right)+2nq^{\frac{n+1}{2}}$$

$$CP(G:q) = 3nq^{2n}+4n\left(q^{2n+1}+q^{2n+2}+...+q^{\frac{5n-3}{2}}\right)+2nq^{\frac{5n-1}{2}}$$

Case (ii) : when n is even and $n \geq 4$
The graph P_2 x C_4 is illustrated in fig (ix)
WDM(P_2 x C_4)



	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	0	7	6	7	7	6	7	6
w_2	1	0	7	6	6	7	6	7
w_3	2	1	0	7	7	6	7	6
w_4	1	2	1	0	6	7	6	7
w_5	1	2	3	2	0	7	6	7
w_6	2	1	2	3	1	0	7	6
w_7	3	2	1	2	2	1	0	7
w_8	2	3	2	1	1	2	1	0

Fig.9 - P_2 x C_4

$DP(P_2 \times C_4:q) = 12q^6 + 16q^7$; DI = 184
$WP(P_2 \times C_4:q) = 12q + 12q^2 + 4q^3$; WI = 48
$CP(P_2 \times C_4:q) = 24q^8 + 4q^{10}$; CI = 232
The graph P_2 x C_6 is illustrated in fig (x)

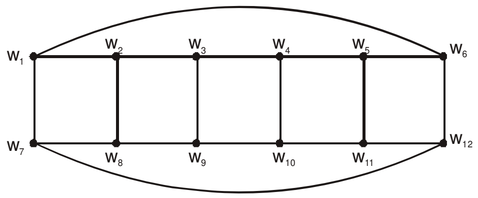


Fig.10 - P_2 x C_6

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}
w_1	0	11	10	11	10	11	11	10	11	10	11	10
w_2	1	0	11	10	11	10	10	11	10	11	10	11
w_3	2	1	0	11	10	11	11	10	11	10	11	10
w_4	3	2	1	0	11	10	10	11	10	11	10	11
w_5	2	3	2	1	0	11	11	10	11	10	11	10
w_6	1	2	3	2	1	0	10	11	10	11	10	11
w_7	1	2	3	4	3	2	0	11	10	11	10	11
w_8	2	1	2	3	4	3	1	0	11	10	11	10
w_9	3	2	1	2	3	4	2	1	0	11	10	11
w_{10}	4	3	2	1	2	3	3	2	1	0	11	10
w_{11}	3	4	3	2	1	2	2	3	2	1	0	11
w_{12}	2	3	4	3	2	1	1	2	3	2	1	0

WDM(P_2 x C_6)

$DP(P_2 \times C_6:q) = 30q^{10} + 36q^{11}$; DI = 696
$WP(P_2 \times C_6:q) = 18q + 24q^2 + 18q^3 + 6q^4$; WI = 144
$CP(P_2 \times C_6:q) = 42q^{12} + 24q^{14}$; CI = 840
Therefore when n is even and $n \geq 4$

$$DP(G:q) = n(n-1)q^{2n-2} + n^2q^{2n-1}$$

$$WP(G:q) = 3nq + 4n\left(q^2 + q^3 + \ldots + q^{\frac{n-2}{2}}\right) + 3nq^{\frac{n}{2}} + nq^{\frac{n+2}{2}}$$

$$CP(G:q) = \left(\frac{3n^2+6n+24}{4}\right)q^{2n} + \left(\frac{3n^2-10n}{2}\right)q^{2n+2}$$

4. CONCLUSION

Thus in this paper we have derived the circular index for some standard graphs and Cartesian product graph P_2 x C_n. This type of circular index plays an important role

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in the movement and storage of products from one place to another. The graph theoretical model of transportation problem helps to find the longest distances and the shortest distances, which minimize the time and cost of round trip of good's carrier.

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