QUOTIENT MODULES DEFINED BY FINITELY GENERATED SUBMODULES OF *QTAG-MODULES*

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Abstract: A right module M over an associative ring with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. Singh, Khan, Mehdi, Abbasi etc. worked a lot on this module. Yet there is much to explore. Here we study the properties shared by QTAG-modules and their quotient modules, where they are defined by finitely generated modules. We find that for a finitely generated submodule N of a QTAG-module M, M is a direct sum of uniserial modules if and only if M/N is a direct sum of uniserial modules. M is Σ module if and only if M/N is a Σ-module. M is summable of countable length if and only if M/N is summable of countable length. A reduced QTAG-modules M of countable length σ is summable if and only if there exists an ascending chain $0 \subseteq S_1 \subseteq \ldots\ldots\ldots\ldots \subseteq S_k$ ***of h-finite submodules of Soc(M) such that*** $\bigcup_{k<\omega} S_k = Soc(M)$.

Keywords: QTAG-modules, Σ-modules, Summable modules

1. INTRODUCTION

All rings R considered here are associative with unity and modules M are unital QTAG-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, d(M) denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\{ d\left(\frac{yR}{xR}\right) \mid y \in M, x \in yR \text{ and } y \text{ uniform}\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k if $k < \omega$ and for the limit ordinal σ, $H_\sigma(M) = \bigcap_{\rho<\sigma} H_\rho(M)$. The generalized height of any $x \in M$ is k if $x \in H_k(M)$, $x \notin H_{k+1}(M)$ here $k<\omega$ or k is a limit ordinal. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_\omega(M)$ and it is h-reduced if it does not contain any h-divisible submodule. M is separable if any submodule $N= \Sigma x_iR$ can be embedded in a direct summand K of M such that K is a direct sum of uniserial modules. A submodule $N \subseteq M$ is high if it is maximal submodule of M such that $N \cap M^1 = 0$. M is a Σ-module if its high submodules are the direct sum of uniserial modules. The cardinality of the minimal generated set of M is denoted by g(M). For a module M there is a chain of submodules $M = M^0 \supset M^1 \supset M^2 \supset \ldots\ldots \supset M^\sigma = 0$ for some ordinal σ. Here $M^{\tau+1} = (M^\tau)^1$ and σ is the height of M. a submodule $N \subseteq M$ is nice in M if $H_\rho(M/N) = (H_\sigma(M)+N)/N$. A

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QTAG-module M of length σ is a summable is Soc(M) = $\oplus M_\beta$ and every nonzero x $\in M_\beta$, if contained in $H_\beta(M)$ such that x $\notin H_{\beta+1}(M)$ for all $\beta < \sigma$.

2. MAIN RESULTS

We start with the following.

Proposition (2.1): Let N be a finitely generated submodules of the QTAG-modules M. Then there exists a finitely generated submodules K $\subseteq$ M such that Soc(M/N) = (K+Soc(M))/N, where N $\subseteq$ K and $H_1(K) \subseteq$ N.

Proof: For $\bar{x} \in$ Soc(M/N), $\bar{x}$ = x + N, where x $\in$ M and there exists y $\in$ N such that H(x)+1 = H(y). Since N is finitely generated, N $\cap$ $H_1(M)$, is also finitely generated such that N $\cap$ $H_1(M)$ = $\sum_{i=1}^{n} z_i R$. As z_i's are in $H^1(M)$, there exists u_i's $\in$ M such that $d(u_iR/z_iR) = 1$, i = 1,……….n. If $u_j \in$ M such that j $\neq$ 1,2,…..n and there exist $z_j \in$ N such that $d(u_jR/z_jR) = 1$, $z_jR \subseteq N$, then $z_jR = z_iR$ for some i, $1 \leq i \leq n$, Therefore $u_jR \subseteq u_iR$ + Soc(M) $\subseteq$ K + Soc(M). The converse is trivial.

Proposition (2.2): Let K be a finitely generated submodule of a QTAG-module M. If N is a nice submodule of M, K $\subset$ L, a submodule of M, then for any submodule Q of M, the following hold:

(i) For x $\in$ Q, there exists y $\in$ M such that x + N = y + N and $H_M(y) = H_{M/N}(y + N)$.

(ii) If that set of heights of the elements of Q in M is finite, then the set of heights of the elements of Q+K is again finite.

(iii) If the set of finite heights of the elements of Q in M is finite, the then the set of finite heights of the elements of Q+K is again finite.

(iv) If the length of M is a limit ordinal and the heights of the elements of Q are bounded, then the heights of Q+K are also bounded.

(v) If Q is a nice in M, then Q+K is nice in M.

(vi) Let P be a submodule of M, containing K. Then P is with a finite number of heights in M if and only if P/K is with a finite number of heights in M/K.

Proof: (i) Let x +N $\in$ M/N such that (x+N) $\in H_\alpha(M/N)$ and (x+N) $\notin H_{\alpha+1}(M/N)$. In other words the height of x + N in M/N is exactly α. This implies that (x+N) $\in$ $(H_\alpha(M)+N)/N$ and (x+N) $\notin$ $(H_{\alpha+1}(M)+N)/N$. Therefore x + N = u + N for some u $\in H_\alpha(M)$ such that u $\notin H_{\alpha+1}(M)$ and the result follows.

(ii) Let A be the set of heights of the elements of Q in M. Here A is the set of ordinals, finite or limit ordinals. Since K is finitely generated, $K = \sum_{i=1}^{n} x_i R$. Out of these $x_1, x_2,$………….. x_n, we may select $x'_1, x'_2,$……….x'_k such that $H(x'_j + y_j) \notin$ A, $1 \leq j \leq k$ and y_j is an element of Q.

Now the set A' = $\{H_M(x'_j + y'_j)\}$ is again finite. If such x'$_j$' do not exist then the result follows, otherwise for any $l \neq 1$, we have $x'_1 + y_l = y_l - y_1 + y_1 + x'_1$. Now $H_M(y_l + x'_1) = \min\{H_M(y_l - y_1), H_M(y_1 + x'_1)\} \in A \cup A'$. Since A, A' both are finite, $A \cup A'$ is also finite and the set of heights of the elements of Q+K is finite.

(iii) Since K is finitely generated, $K = \sum_{i=1}^{n} x_i R$. Now the heights of the elements of Q are finite and A $\{H_M(x) | x \in Q\}$ is also finite, therefore there exists some m such that $Q \cap H_m(M) \subseteq H_\omega(M)$. Let $z_j \in Q$, $y_j \in K$ such that $z_j + y_j \in H_{t_j}(M)$, $z_j + y_j \notin H_{t_{j+1}}(M)$, $1 \leq j \leq k$ and $z_{k+1} + y_{k+1}, \ldots\ldots\ldots, z_n + y_n \in H_\omega(M)$ for some integer t_1, ……. $t_k \geq m$, $k \leq n$.

Now every elements of Q+K is of the form $z+y = z + y_i$ where either $1 \leq i \leq k$ or $k+1 \leq i \leq n$. Again $z + y_i = z - z_i + z_i + y_i \notin H_{t_{i+1}}(M)$, if $x+y_i \notin H_\omega(M)$, otherwise $z - z_i + y_i + z_i \in H_{t_i}(M)$ implying that $z - z_i \in H_{t_i}(M)$ and $z - z_i \in H_\omega(M)$.

This means $z_i + y_i \in H_{t_{i+1}}(M)$, which is a contradiction when $1 \leq i \leq k$ or $k + 1 \leq i \leq n$.

Therefore $z_i + y_i \in H_\omega(M)$ and thus $z + y_i \in H_\omega(M)$. If we put $l = \max(t_{i+1}, \ldots\ldots\ldots t_{k+1})$, then $l \geq m$ and $(Q+K) \cap H_l(M) \subseteq H_\omega(M)$, proving the results.

(iv) Let the length of M be σ. Since the heights of the elements of Q are bounded, there exists and ordinal $\rho < \sigma$ such that $Q \cap H_\rho(M) = 0$. Now the method adopted in the proof of (iii), ensures the existence of an ordinal $\beta < \sigma$, such that $(Q+k) \cap H_\beta(M) = 0$.

(v) Let β be an arbitrary limit ordinal. Since K is finitely generated, $\bigcap_{\alpha<\beta}(H_\alpha(M)+Q+K) = \bigcap_{\alpha<\beta}((H_\alpha(M)+Q)+K \; and \; \bigcap_{\alpha<\beta}(H_\alpha(M)+Q+K = H_\beta(M)+Q+K$. Therefore Q+K is nice in M.

(vi) Since K is finitely generated, it is nice and for every $x+k \in P/K$, $x \in P$, we have $x+K = y+K$ for some $y \in M$ such that $H_{M/K}(x+K) = H_M(y) = H_{M/K}(y+K)$ as $y \in P+K$. Now by asserting (ii) the necessity follows.

For the converse, for each $x \in P$, $x + K = y + K$, where $H_M(y) = H_{M/K}(x+K) = H_{M/K}(y+K)$. Therefore $x = y + z$ for some $z \in K$. Now $H_M(x) \leq H_{M/K}(x+K) = H_M(y)$. If $H_M(y) > H_M(z)$, then $H_M(x) = H_M(z)$ and the result follows. If $H_M(y) \leq H_M(z)$, then $H_M(x) = H_M(y) = H_{M/K}(x+K)$ and we are through, otherwise $H_M(y) = H_M(z)$ and $H_M(x) \geq H_M(y) = H_{M/K}(x+K) > H_M(x)$. This implies that $H_M(x) = H_{M/K}(x+K)$ and the result follows.

Now we are able to prove the following Theorem.

Theorem (2.3): Let N be a finitely generated submodule of a separable module M. Then M is a direct sum of uniserial modules if and only if M/N is a direct sum of unisirial modules.

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Proof: Let M be a direct sum of uniserial modules. By [3] we may write M = $\bigcup_{k<\omega} M_k$, $M_k \subseteq M_{k+1} \subseteq M$ with $M_k \cap H_k(M) = 0$. Therefore M/K = $\bigcup_{k<\omega}\left(\frac{M_k + N}{N}\right)$. By Proposition 2.2 (iii), there exists a natural number $m_k > k$, such that $(M_k+N) \cap Hm_k(M) = 0$. Since M is separable, $\left(\frac{M_k + N}{N}\right) \cap Hm_k(M/N)$ = $[(M_k+N) \cap (Hm_k(M)+N)]/N = [N + (M_k+N) \cap Hm_k(M)]/N$. Now by [2], M/N is a direct sum of uniserial modules.

Conversely, M/N may be expressed as $\bigcup_{k<\omega}(P_k / N) = \left(\bigcup P_k\right)/ N$, where $P_k \subseteq P_{k+1} \subseteq M$ such that $H_k(M/N) \cap (P_k/N) = 0$, for every index k. As $P_k \cap H_k(M) \subseteq N$, N is finitely generated and M is separable, we have $N \cap H_n(M) = 0$, for some $n < \omega$. This implies that $P_k \cap H_{k+n}(M) = 0$. Since M = $\bigcup_{k<\omega} P_k$, by [2], M is a direct sum of uniserial module.

Now we characterize some QTAG-modules namely Σ-modules, summable modules, σ-modules etc.

Theorem (2.4): Let N be a finitely generated submodule of a module M. Then M is a Σ-modules if and only if M/N is a Σ-module.

Proof: Let N be a finitely generated submdoules of a Σ-module. Now Soc(M) = $\bigcup_{k<\omega} S_k$, $S_K \subseteq S_{k+1} \subseteq Soc(M)$ and $S_k \cap H_k(M) \subseteq H_\omega(M)$. Now by Proposition 2.1, Soc(M/N) = (Soc(M) + K)/N for some finitely generated submodule $K \subseteq M$, containing N s.t $N \subseteq K$, $H_1(K) \subseteq N$ and $\left(\frac{Soc(M)+K}{N}\right) = \bigcup_{k<\omega}\left(\frac{S_k + K}{N}\right)$. Now by Proposition 2.2 (iii) for every integer $k \geq 1$, there exists an integer $n_k \geq k$ such that $(S_k+K) \cap Hn_k(M) \subseteq H_\omega(M)$. Since $\left(\frac{S_k + K}{N}\right) \cap Hn_k(M/N)$ =

$$\frac{(S_k + K)\bigcap(Hn_k(M)+N)}{N} = \frac{N+(S_k + K)\bigcap Hn_k(M)}{N} \subseteq \frac{N + H_\omega(M)}{N} = H_\omega(M/N).$$

Therefore by [2], M/N is a Σ-module.

For the converse, we may write Soc(M/N) = $\bigcup_{k<\omega}\left(\frac{S_k}{N}\right)$ with $S_k \subseteq S_{k+1} \subseteq M$ and $\left(\frac{S_k}{N}\right) \cap H_k(M/N) = H_\omega(M/N)$. Since N is finitely generated hence nice in M, $\left(\frac{S_k}{N}\right) \cap H_k(M/N) = \frac{[S_k \bigcap (H_k(M)+N)]}{N} = \frac{H_\omega(M)+N}{N}$. Now there exists $n_k \geq n$, such that $N \cap Hn_k(M) \subseteq H_\omega(M)$ and we have $S_k \cap Hn_k(M) \subseteq (H_\omega(M)+N) \cap Hn_k(M) = H_\omega(M)$

+ (N $\cap$ Hn_k(M)) = H_ω(M). Since $\left(\frac{Soc(M)+N}{N}\right)$ $\subseteq$ Soc(M/N), Soc(M) = $\bigcup_{k<\omega} Soc(S_k)$ and again by [2], M is a Σ-module.

Remark (2.5): If N is a finitely generated submodule of M such that N $\cap$ H_ω(M) = 0, then N is nice in M and for a high submodule K/N of M/N, $\left(\frac{K}{N}\right)\cap H_\omega(M/N)=\left(\frac{M}{N}\right)\cap\left(\frac{H_\omega(M)+N}{N}\right)=\frac{K\cap H_\omega(M)+N}{N}=0$. Now is M is a Σ-module, then K $\subseteq$ K', a high submodule of M and K' is the direct sum of uniserial module. Therefore, by Theorem 2.3, K/N is a direct sum of uniserial modules. Converse is trivial.

To investigate summable module we start with the following.
Definition (2.6): A submodule N of M is h-finite if the height of the elements of M assure but a finite number of different values.

Remark (2.7): For a reduced QTAG-modules M of length σ, we define S_ρ such that Soc(H_ρ(M)) = Soc($H_{\rho+1}$(M)) ⊕ S_ρ, $\rho < \sigma$. Therefore the nonzero elements of S_ρ are of heights ρ. We may consider $\bigoplus_{\rho<\sigma} S_\rho, \rho<\sigma$. Therefore the nonzero elements of S_ρ are of height ρ. We may consider $\bigoplus_{\rho<\sigma} S_\rho$ and if x = $x_{\rho 1}$ + …… + $x_{\rho k}$, 0 ≠ $x_{\rho i}$ ∈ $S_{\rho i}$, then the generated height of x is min(ρ_1………ρ_k).

Proposition (2.8): A reduced QTAG – modules M of countable length σ is summable if and only if there exists an ascending chain 0 $\subseteq$ S_1 $\subseteq$……$\subseteq$ S_k of h-finite submodules of Soc(M) such $\bigcup_{k<\omega} S_k = Soc(M)$.

Proof: Let M be summable and σ-countable. Then we can define S_ρ, for $\rho < \sigma$ and form a sequence $S_{\rho 1}$, $S_{\rho 2}$,…….,$S_{\rho k}$. Now S_k = $S_{\rho 1}$⊕ ……⊕$S_{\rho k}$ are h-finite and $\bigcup_{k<\omega} S_k = Soc(M)$.

For the converse, consider the ascending chain of h-finite submodules S_k of M such that $\bigcup_{k<\omega} S_k = Soc(M)$. As H_σ(M) $\cap$ S_1 is a summand of S_1 and if ρ_1 is the maximal height of the nonzero elements is S_1, we may express S_1 = $N_{\rho 1}$⊕ ……… $N_{\rho n}$ where $\rho_1 > \rho_2 >$ ………. > ρ_n and every nonzero elements of $N_{\rho i}$ has height ρ_i.

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Inductively for every k we can construct $S_k = \bigoplus_{\rho<\sigma} N_\rho^{(k)}$ such that $N_\rho^{(k)}$ =0 for almost all $\rho, N_\rho^{(k)} \subseteq H_\rho(M), N_\rho^{(k)} \cap H_{\rho+1}(M) = 0$ and $N_\rho^{(k-1)} \subseteq N_\rho^{(k)}$. By putting $S_\rho = \bigcup_k N_\rho^{(k)}$, we get Soc(M) = $\bigoplus_{\rho<\sigma} S_\rho$ i.e. M is summable.

Theorem (2.9): Let N be finitely generated submodule of M ,the QTAG-module. Then M is summable of countable length if and only if M/N is summable of countable length.

Proof: Let length of M is less than ω+1 if and only if the length of M/N is less than ω+1. Suppose M is summable and Soc(M) = $\bigcup_{k<\omega} S_k \subseteq S_{k+1} \subseteq$ Soc(M) and $S_k^{'}s$ are h-finite in M. By Proposition 2.1, Soc(M/N) = $\left(\frac{Soc(M)+K}{N}\right)$ for some finitely generated K ⊆ M such that $H_1(K) \subseteq$ N. Now Soc(M/N) = $\bigcup_{k<\omega}\left(\frac{S_k+K}{N}\right)$ Again by Proposition 2.2(i) and (iv) we find $\left(\frac{S_k+K}{N}\right)$ are h-finite in M/N. Therefore by proposition 2.8 M/N is summable. For the converse Soc(M/N) = $\bigcup_{k<\omega}\left(\frac{T_k}{N}\right) = \frac{\bigcup T_k}{N}$ such that $T_K \subseteq T_{k+1} \subseteq$ M and T_k/N are h-finite in M/N. Since $\frac{Soc(M)+N}{N} \subseteq Soc(M/N)$. We may infer that Soc(M) = $\bigcup Soc(T_k)$. Now by Proposition 2.2 (vi) we may conclude that $T_k^{'}s$ are h-finite in M, thus $Soc(T_K)'s$ are also h-finite in M. Therefore M is summable.

3. REFERENCES

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