SUM LABELING FOR CYCLE SPARKLE GRAPH & ARBITRARY SUPERSUBDIVISION OF TREE

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Abstract: A Sum Labeling is a mapping λ from the vertices of G into the positive integers such that, for any two vertices u, v ϵ V (G) with labels $\lambda(u)$ and $\lambda(v)$, respectively, (uv) is an edge iff $\lambda(u) + \lambda(v)$ is the label of another vertex in V(G). Any graph supporting such a labeling is called a Sum graph. A set of isolated vertices known as isolates are added (as a disjoint union) to sum label a graph and the labeling scheme that requires the fewest isolates is termed optimal. The number of isolates required for a graph to support a sum labeling is known as the sum number of the graph.

In this paper, we introduce a new graph called cycle sparkle graph and obtain optimal sum labeling scheme for cycle sparkle graph and arbitrary supersubdivision of tree.

Keywords: Sum Labeling, Sum number, Sum graph, Arbitrary Supersubdivision 2010 AMS Subject Classification: 05C78

1. INTRODUCTION

All the graphs considered here are simple, finite and undirected. For all terminologies and notations we follow Harary [5] and graph labeling as in [3]. Sum labeling of graphs was introduced by Harary [6] in 1990. Following definitions are useful for the present study.

Definition 1.1 A ***Sum Labeling*** is a mapping λ from the vertices of G into the positive integers such that, for any two vertices u, v $\in$ V(G) with labels $\lambda(u)$ and $\lambda(v)$, respectively, (uv) is an edge iff $\lambda(u) + \lambda(v)$ is the label of another vertex in V (G). Any graph supporting such a labeling is called a ***Sum Graph***.

Definition 1.2 To sum label a graph adding a component (as a disjoint union) is necessary. This disconnected component is a set of isolated vertices known as ***Isolates*** and the labeling scheme that requires the fewest isolates is termed ***Optimal.***

Definition 1.3 The number of isolates required for a graph G to support a sum labeling is known as the ***Sum Number*** of the graph. It is denoted as $\sigma(G) = 1$.

Definition 1.4 [1] A tree is called a ***spider*** if it has a center vertex c of degree k > 1 and each other vertex either is a leaf (pendent vertex) or has degree 2. Thus a spider is an amalgamation of k paths with various lengths. If it has x_1 paths of length a_1, x_2 paths of length a_2, ..., x_n paths of length a_m, we denote the spider by SP($a_1^{x_1}, a_2^{x_2}, \ldots, a_m^{x_n}$) where $a_1 < a_2 < \ldots < a_m$ and $x_1 + x_2 + \ldots + x_n = k$

Definition 1.5 Cycle Sparkle Graph is the graph obtained from a cycle of *p* vertices and *p* spiders (not necessarily unique) by joining the centre vertex of the i^th^ spider with the i^th^ vertex of the cycle**.** It is denoted by $\mathbf{CSG_p}$

Definition 1.6 Let G be a graph with q edges. A graph H is called a ***Super subdivision*** of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \le i \le q$ in such a way that the end vertices of each e_i are identified with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G. If m_i is varying arbitrarily for each edge e_i then super subdivision is called ***Arbitrary Super subdivision*** of G.

In this paper, we will prove that cycle sparkle graph is sum graph with $\sigma(G) = 1$ and graph obtained by arbitrary super subdivision of tree is optimal summable with $\sigma(G) = 2$.

2. OPTIMAL SUM LABELING SCHEME FOR CYCLE SPARKLE GRAPHS

Lawrence Rozario and Gerard Rozario [8], introduced Flower Pot Cracker graph ($FPC_{p,t}$) and proved that $FPC_{p,t}$ for all $p > 3$ & $t > 1$ is a combination graph. In [4], Gerard Rozario et.al proved that Flower Pot Cracker graph $FPC_{p,t}$ is sum graph with sum number 1 for all $p \ge 3$ and $t > 1$.

In this section, we prove that cycle sparkle graph is sum graph with $\sigma(G) = 1$.

Theorem 2.1: Cycle Sparkle Graph (CSG_p) is sum graph with $\sigma(G) = 1$.

Proof: Let G be cycle sparkle graph with cycle of *p* vertices and *p* spiders. Let n_1, n_2, n_3, ……, n_p be the number of paths in 1^{st}, 2^{nd}, 3^{rd},……, p^{th} vertex respectively. Let k be total number of paths. i.e., $k = n_1 + n_2 + n_3 + \ldots\ldots + n_p$. Let m_i be the length of i^{th} path where $1 \le i \le k$. Let c_1, c_2, c_3,…, c_p be the vertices of the cycle. Let $n_0 = 0$. Let $v_{11}, v_{12}, \ldots, v_{1m_1}, v_{21}, v_{22}, \ldots, v_{2m_2}, \ldots\ldots, v_{k1}, v_{k2}, \ldots, v_{km_k}$ be the vertices of 1^{st}, 2^{nd},……, k^{th} paths respectively. Let x be the isolated vertex. Define $f : V(G) \to N$

The vertices of cycle (C_p) and the vertex (v_{11}) are labeled based on the value of p as discussed below in various cases.

Case(i) If p is odd number

Case (i)a: p = 3

$f(c_1) = 1$ $\quad$ $f(c_2) = 2$ $\quad$ $f(c_3) = 3$

$f(v_{11}) = f(c_1) + f(c_3)$

Case (i)b: p = 5, 7

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$f(c_1)=2 \;\; f(c_2)=1 \qquad f(c_3)=3$

$f(c_p)=f(c_1)+f(c_3)$

$f(c_i)=f(c_{(i-1)})+f(c_{(i-2)})$ for $4\le i\le (p-1)$

$$f(v_{11})=\begin{cases} f(c_p)+f(c_{(p-1)}) & \text{if } f(c_p)<f(c_{(p-1)})\\ f(c_{(p-1)})+f(c_{(p-2)}) & \text{if } f(c_p)>f(c_{(p-1)})\end{cases}$$

Case(i)c:p ≥ 9

$$f\left(c_{\left(\frac{p-3}{2}\right)}\right)=1,\; f\left(c_{\left(\frac{p-3}{2}-1\right)}\right)=2,\; f\left(c_{\left(\frac{p-3}{2}+1\right)}\right)=3,\; f\left(c_{\left(\frac{p-3}{2}+2\right)}\right)=4,$$

$$f\left(c_{\left(\frac{p-3}{2}-2\right)}\right)=5 \quad \text{for } 1\le i\le\left(\frac{p-9}{2}\right)\begin{cases} f\left(c_{\left(\frac{p-3}{2}+i+2\right)}\right)=f\left(c_{\left(\frac{p-3}{2}-i\right)}\right)+f\left(c_{\left(\frac{p-3}{2}-i-1\right)}\right)\\ f\left(c_{\left(\frac{p-3}{2}-i-2\right)}\right)=f\left(c_{\left(\frac{p-3}{2}+i+1\right)}\right)+f\left(c_{\left(\frac{p-3}{2}+i+2\right)}\right)\end{cases}$$

$f(c_{(p-3)})=f(c_1)+f(c_2),\, f(c_p)=f(c_{(p-3)})+f(c_{(p-4)})$

$f(c_{(p-2)})=f(c_1)+f(c_p)$

$f(c_{(p-1)})=f(c_{(p-3)})+f(c_{(p-2)})$

$f(v_{11})=f(c_p)+f(c_{(p-1)})$

Case(ii): If p is even number

Case (ii)a: p = 4

$f(c_1)=1 \qquad f(c_2)=2 \qquad f(c_4)=3 \;\; f(c_3)=f(c_1)+f(c_4)$

$f(v_{11})=f(c_2)+f(c_3)$

Case (ii)b: p ≥ 6

$f(c_1)=1 \qquad f(c_2)=2 \qquad f(c_p)=3 \;\; f(c_3)=f(c_2)+f(c_p)$

$f(c_{p-1})=f(c_1)+f(c_p)$

for $4 \le i \le p/2$

$$\begin{cases} \text{if } i \text{ is even}, \begin{cases} f(c_i) = f(c_{i-1}) + f(c_{i-2}) - 1 \\ f(c_{i+(p-2i+2)}) = f(c_{i+(p-2i+2+1)}) + f(c_{i+(p-2i+2)+2}) \end{cases} \\ \text{if } i \text{ is odd}, \begin{cases} f(c_i) = f(c_{i-1}) + f(c_{i-2}) \\ f(c_{i+(p-2i+2)}) = f(c_{i+(p-2i+2+1)}) + f(c_{i+(p-2i+2)+2}) - 1 \end{cases} \end{cases}$$

$$f\left(c_{\left(\frac{p}{2}+1\right)}\right) = f\left(c_{\left(\frac{p}{2}\right)}\right) + f\left(c_{\left(\frac{p}{2}-1\right)}\right)$$

$$f(v_{11}) = \begin{cases} f\left(c_{\left(\frac{p}{2}\right)}\right) + f\left(c_{\left(\frac{p}{2}+1\right)}\right) & \text{if} \quad f\left(c_{\left(\frac{p}{2}\right)}\right) < f\left(c_{\left(\frac{p}{2}+2\right)}\right) \\ f\left(c_{\left(\frac{p}{2}+1\right)}\right) + f\left(c_{\left(\frac{p}{2}+2\right)}\right) & \text{if} \quad f\left(c_{\left(\frac{p}{2}\right)}\right) > f\left(c_{\left(\frac{p}{2}+2\right)}\right) \end{cases}$$

Thus, the vertices of cycle (C_p) and the vertex (v_{11}) are labeled. All the other vertices, namely $v_{12},\dots,v_{1m_1}, v_{21}, v_{22},\dots,v_{2m_2},\dots\dots, v_{k1}, v_{k2},\dots,v_{km_k}$ of spiders in all cases discussed above follows the following logic.

$$\text{for } 1 \le s \le p \quad \begin{cases} \text{for } \left(\sum_{i=1}^{s} n_{(i-1)}\right) + 1 \le i \le \left(\sum_{i=1}^{s} n_i\right) \\ \begin{cases} \text{if } m_i > 1 \\ \begin{cases} \begin{cases} f(v_{ij}) = f(c_s) + f(v_{i1}) & ;\ j = 2 \\ f(v_{ij}) = f(v_{i(j-1)}) + f(v_{i(j-2)}) & \text{for } 3 \le j \le m_i \end{cases} \\ f(v_{(i+1)1}) = f(v_{ij}) + f(v_{i(j-1)}) \quad \text{if } j = m_i \text{ and } i \ne k \end{cases} \\ \text{if } m_i = 1 \text{ and } i \ne k \\ f(v_{(i+1)1}) = f(v_{i1}) + f(c_s) \end{cases} \end{cases}$$

$$f(x) = \begin{cases} f(v_{km_k}) + f(v_{k(m_k-1)}) & \text{if } m_k > 1 \\ f(v_{km_k}) + f(c_p) & \text{if } m_k = 1 \end{cases}$$

Thus, Cycle Sparkle Graph is sum graph with $\sigma(G) = 1$ for all $p \ge 3$.

Illustration 2.1: *Sum labeling for Cycle Sparkle Graph* (*CSG_5*) with *cycle of 5 vertices is shown in figure 2.1*

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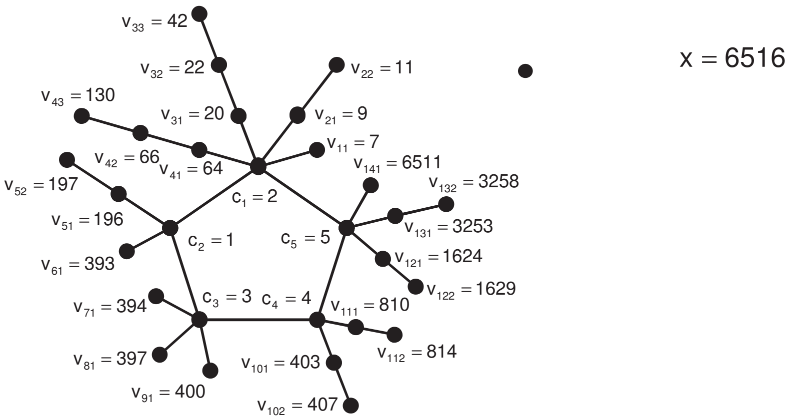

Figure 2.1

3. OPTIMAL SUM LABELING FOR ARBITRARY SUPER SUBDIVISION OF TREE

Sethuraman et.al [10], introduced a new method of construction called Supersubdivision of graph and proved that arbitrary supersubdivision of any path and cycle C_n are graceful. Kathiresan et.al [7], proved that arbitrary supersubdivision of any star is graceful. In [4], Gerard Rozario et.al proved that the arbitrary super subdivision of path P_n, cycle C_n and star $K_{1,n}$ are optimal summable with sum number 2. In [2], Dani Nilish proved that arbitrary supersubdivision of tree is cordial. In [9], Miller et.al, gave working draft for sum labeling of trees.

In this section, we prove that the arbitrary super subdivision of tree is optimal summable with $\sigma(G) = 2$.

Theorem 3.1 *Arbitrary supersubdivision of tree T of order ≥ 2 are optimal summable with $\sigma(G) = 2$.*

Proof: Let T be a tree of order ≥ 2 with n vertices. Let v_i $(1 \leq i \leq n)$ be the vertices of T. The vertices of T are identified using Modified Depth First Search (MDFS) algorithm [9]. The algorithm follows the Depth First Search (DFS) algorithm with one modification. The MDFS is just one of the possible implementation of DFS algorithm in which edges with vertices of degree 1 are given higher priority.

The algorithm is as follows: Whenever the algorithm finds a new vertex $x \in V(T)$ of a tree T having degree ≥ 3, it first visits all its neighbors of degree 1 (if any), before proceeding to branches of T at x of higher order. The algorithm, for ordering of vertices, follows the MDFS starting with $v_1, v_2, \ldots\ldots, v_n$.

Let T* be the arbitrary supersubdivision of T which is obtained by replacing every edge of T with K_{2,m_i} . Let $m = \sum_{i=1}^{n-1} m_i$. Let u_j be the vertices which are used for arbitrary supersubdivision of T where $1 \le j \le m$. Let x and y be two isolated vertices. Therefore, the vertex set of T* is V(T*) = { v_1, $v_2, \ldots\ldots, v_n, u_1, u_2, \ldots\ldots, u_m$}.Define $f : V(T^*) \to N$

$$f(v_i) = i \text{ for } 1 \le i \le n,\ f(u_1) = m + n$$

$$f(u_j) = f(u_{j-1}) - 1 \text{ for } 2 \le j \le m$$

$$f(x) = f(u_1) + 1 \quad \text{and} \quad f(y) = f(u_1) + 2$$

Hence, arbitrary supersubdivision of a tree of order ≥ 2 are optimal summable with $\sigma(G) = 2$.

Illustration 3.1: *Sum labeling of arbitrary supersubdivision of tree T of 9 vertices is shown in figure 3.1.*

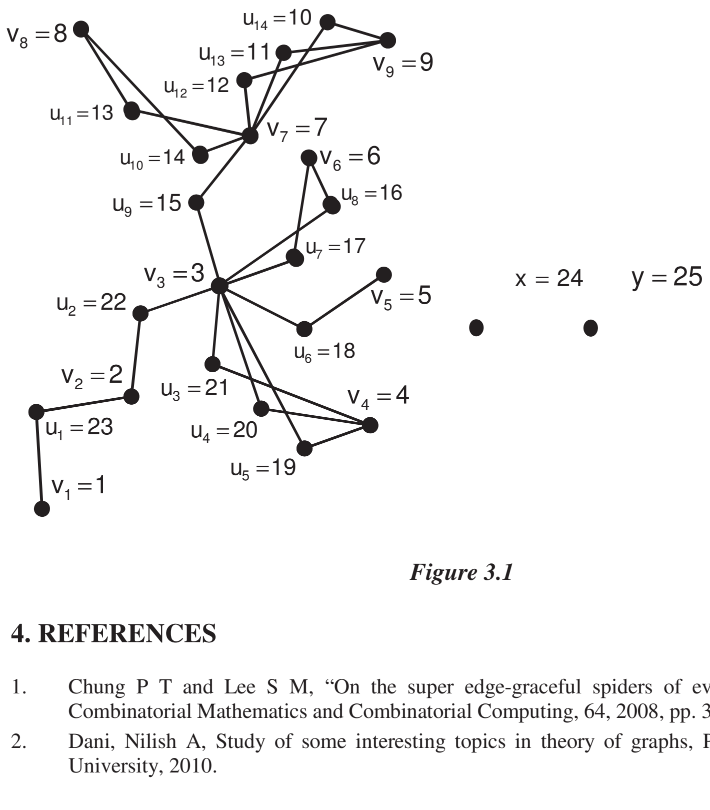

Figure 3.1

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