APPLICATION OF BETA, GAMMA AND PSI FUNCITIONS IN SEDIMENT TRANSPORT

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Abstract: Analytical approximations are proposed using beta, gamma and psi functions to calculate suspended-load transport rate in terms of fast-converging series. Proposed approximations of the integrals related to suspended-load transport rate, agree well with corresponding numerical solution for all values of their arguments despite of the simplifications used. Finally suspended-load transport rate in river and laboratory experiment is calculated using these analytic approximations and the agreement with the numerical result and measured data is good.

Keywords: Suspended-load transport rate, velocity and concentration distribution, logarithmic law, Rouse number, Taylor series.

1. INTRODUCTION

The calculation of sediment transport rate is one of the most important aspects in many coastal engineering studies. Over the past hundred-plus years, much effort has been given to predict the sediment transport rate. Numerous investigations have been done and many procedures have been proposed for prediction of the sediment transport rate. Sediment transport is generally classified into two categories: bed-load and suspended-load. In many natural flows the suspended sediment transport dominates the total transport rate. Measuring the suspended-load transport rate is simpler than the bed-load transport rate at least in principle. Among many procedures, depth-integrand sampling method is usually used. This method requires the estimation of integrals which cannot be integrated in closed form for most of the cases. Many authors have provided numerical techniques [1], [2] to calculate these integrals. Einstein [1] provided a numerical table to facilitate these numerical integrations [3]. Direct numerical integration method is very slowly convergent due to the singularity of the integrand functions near the channel bed [2], [3]. Guo and Wood [3] provided a way to compute Einstein's integrals analytically for fine sediments (fine sediment is defined as the Rouse number $Z_1 < 1$) using beta, gamma and psi functions. Guo [4] proposed good approximation for gamma and psi functions and used them to calculate suspended load transport rate. Later on, [5] provided an algorithm to compute Einstein's integrals for all values of the Rouse number. The integrand function appears in Einstein integrals [4] results after assuming a von Karman-Prandtl's type velocity profile and Rouse's concentration profile. Rouse equation was derived assuming a parabolic type eddy viscosity profile [3] and the effect of secondary circulation was neglected [6]. Many attempts are made toward the modification of the Rouse equation. Considering the effect of secondary current, [7] provided an analytical equation for suspended concentration distribution which is applicable for fine, medium and coarse sands. The suspended load transport rate is calculated using the logarithmic velocity profile and a modified Rouse equation.

In this paper we present a method to compute the integrals (for all types of sediments) analytically and propose very effective approximations using beta, gamma and psi functions to these integrals that appear in suspended-load transport rate. We compare our result with the actual numerical result and measured data.

2. BETA, GAMMA AND PSI FUNCTION AND THEIR PROPERTIES:

Beta function $B(x, y)$ is usually defined in integral form as [8, pp. 193]

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad \text{where } x>0,\ y>0 \tag{1}$$

and there are several ways in literature to define the gamma function. According to Weierstrass the gamma function is defined for any real number x except the negative integers $(0,-1,-2,\ldots)$ as [9]

$$\Gamma(x)=\frac{e^{-\gamma x}}{x}\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)^{-1}e^{x/k} \tag{2}$$

where $\gamma = 0.57721566490\ldots$ is the Euler-Mascheroni constant. The most common way to define the gamma function in integral from as [9]

$$\Gamma(x)=\int_0^{\infty} e^{-t}t^{x-1}dt \quad \text{where} \quad x>0 \tag{3}$$

One can show the following relations hold

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4}$$

$$\Gamma(x+1)=x\Gamma(x) \tag{5}$$

$$\Gamma(1-x)\Gamma(1+x)=\frac{\pi x}{\sin \pi x} \text{ where } \quad 0<x<1 \tag{6}$$

The psi or the digamma function is defined as [9]

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)} \tag{7}$$

where x is any non-null or non-negative integer. Differentiating Eq. (5) and using Eq. (7) one can show that psi function follows the following recurrence formula

$$\psi(x+1)=\psi(x)+\frac{1}{x} \tag{8}$$

and with repeated application of Eq. (8) one can show that

$$\psi(x+n)=\psi(x)+\sum_{k=1}^{n}\frac{1}{x+k-1} \tag{9}$$

in which n is any natural number. Beside the definition, one can represent psi function in series expansion by taking logarithm on both side of Eq. (2) and then differentiating with respect to x as

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma-\frac{1}{x}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{x+k}\right) \quad \text{where}\, x>0 \tag{10}$$

The series in Eq. (10) is slowly convergent. It is clear that for any natural number n we have

$$\psi(n)=-\gamma+\sum_{k=1}^{n-1}\frac{1}{k} \tag{11}$$

in particular we have $\psi(1)=-\gamma$ and $\psi(2)=1-\gamma$.

The psi function satisfies the following reflection relation [9]

$$\psi(1-x)=\psi(x)+\pi\cot\pi x \tag{12}$$

Where $x\in \mathrm{R}$ and $x\notin \mathrm{Z}$ in which R and Z are set of real numbers and integers respectively.

The Pochhammer symbol is defined as [10]

$$(\lambda)_n=\begin{cases}1 & n=0,\ \lambda\neq 0\\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & n\in \mathrm{N},\ \lambda\in \mathrm{R}\end{cases} \tag{13}$$

where N is the set of natural numbers.

One useful relation can be obtained by equating Eq. (1) and Eq. (4) and differentiating it with respect to x and using Eq. (7) as follows

$$\int_0^1 t^{x-1}(1-t)^{y-1}\ln t\,dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\{\psi(x)-\psi(x+y)\} \tag{14}$$

The equations (4), (6), (9), (11), (12), (13) and (14) will be used later in this paper to simplify the calculations.

3. APPLICATION IN SUSPENDED-LOAD TRANSPORT RATE

The suspended-load refers to sediment particles that stay in suspension during some period of time in the flow. The suspended-load transport rate which is defined as the mass of sedimentary material that passes across the cross section of an open-channel for a given flow per unit time and is denoted in the integral form as [1]

$$q_s = \int_a^h Cudy \qquad (15)$$

where C is the concentration of sediment particles in suspension, u is the flow velocity in the direction of flow, y represents the vertical co-ordinate, a is the bed-load layer thickness and h is the flow depth. The velocity profile is usually given by the logarithmic law [1], [11], [12] which is as follows

$$\frac{u}{u_*} = \left(\frac{1}{\kappa} + \frac{U}{u_*}\right) + \frac{1}{\kappa}\ln\xi \qquad (16)$$

where u_* is shear velocity, U is average velocity, $\kappa(=0.4)$ is the von Karman coefficient and $\xi(= y/h)$ is a dimensionless quantity corresponding to height. Sediment concentration is usually described by the well-known Rouse equation [13]. It is applicable for such sediment-laden flows where sediment concentration is low. In other words, Rouse equation systematically deviates from measured data for high sediment concentrated flows. According to [7] the suspended concentration is described by a modified Rouse equation as

$$\frac{C}{C_a} = \left(\frac{1-\xi}{\xi}\frac{\xi_a}{1-\xi_a}\right)^{Z_1} \exp\{Z_2(\xi-\xi_a)\} \qquad (17)$$

where C_a is the near bed concentration at height $y = a$, $\xi_a(= a/h)$ is a dimensionless quantity representing the reference level, $Z_1 = \omega/(\kappa\gamma u_*)$ is the Rouse number in which ω is the fall velocity of sediment particle, γ is the proportionality constant act as a parameter and $Z_2 = \alpha/(\kappa\gamma)$ where α is a parameter. Substituting Eq. (16) and (17) into Eq. (15) the suspended-load transport rate can be written as

$$q_s = C_a u_* h\left(\frac{\xi_a}{1-\xi_a}\right)^{Z_1} \exp(-Z_2\xi_a)\int_{\xi_a}^{1}\left(\frac{1-\xi}{\xi}\right)^{Z_1}\exp(Z_2\xi)\left[\left(\frac{1}{\kappa}+\frac{U}{u_*}\right)+\frac{1}{\kappa}\ln\xi\right]d\xi$$

$$= C_a u_* h\left(\frac{\xi_a}{1-\xi_a}\right)^{Z_1} \exp(-Z_2\xi_a)\left[\left(\frac{1}{\kappa}+\frac{U}{u_*}\right)I_1 + \frac{1}{\kappa}I_2\right] \qquad (18)$$

where the integrals I_1 and I_2 are given by

$$I_1 = \int_{\xi_a}^{1}\left(\frac{1-\xi}{\xi}\right)^{Z_1}\exp(Z_2\xi)\,d\xi \qquad (19)$$

and

$$I_2 = \int_{\xi_a}^{1}\left(\frac{1-\xi}{\xi}\right)^{Z_1}\exp(Z_2\xi)\ln\xi\,d\xi \qquad (20)$$

Integrals (19) and (20) are identical with the Einstein integrals [4] except the exponential terms. Integrations I_1 and I_2 can be evaluated using any numerical technique. Guo and Julien [5] provided an efficient algorithm to compute Einstein integrals. In this paper, following them we provide an analytical way to compute these two integrals (19) and (20). One can further write integrals (19) and (20) as follows

$$I_1 = \int_0^1 f(Z_1, Z_2, \xi) d\xi - \int_0^{\xi_a} f(Z_1, Z_2, \xi) d\xi = J_1 - J_2 \quad (21)$$

and

$$I_2 = \int_0^1 g(Z_1, Z_2, \xi) d\xi - \int_0^{\xi_a} g(Z_1, Z_2, \xi) d\xi = J_3 - J_4 \quad (22)$$

where $f(\xi)$ and $g(\xi)$ are integrand functions as in Eq. (19) and (20) respectively and integrals J_1 and J_3 have limits of integration from 0 to 1 and integrals J_2 and J_4 have limits of integration from 0 to ξ_a. For the purpose of simplifying the manipulation, in this study we consider that $Z_2 \le 1$. The bed layer thickness is small relative to the flow depth and for small values of ξ_a, in the interval $[0, \xi_a]$ one can approximate the integrand in integral J_2 and J_4 for any value of Z_1 as

$$f(Z_1, Z_2, \xi) \approx \left(\frac{1-\xi}{\xi}\right)^{Z_1} (1 + Z_2 \xi) \quad (23)$$

A comparison between approximated values and exact values is plotted in Fig. 1 for $\xi_a = 10^{-1}$ and for five different values of the parameter Z_2. From Fig. 1 one can observe that for any value of Z_2 less than one, the integrand in the interval $[0, \xi_a]$ can be approximated by Eq. (23) where the maximum relative error is 0.47% for $0 \le Z_2 \le 1$. In fact it is also true for any subinterval $[0, E]$ of $[0, \xi_a]$ where $E(< \xi_a)$ is any real number. Furthermore if we approximate the exponential function as $\exp(Z_2 \xi) \approx 1$ then integrals I_1 and I_2 takes the special form which is known as Einstein [1] integrals.

3.1 Integral J_1: Using the Taylor series expansion of the exponential function, the integral J_1 can be written as

$$J_1 = \int_0^1 \left(\frac{1-\xi}{\xi}\right)^{Z_1} \left[\sum_{n=0}^{\infty} \frac{Z_2^n}{n!} \xi^n\right] d\xi \quad (24)$$

Using the definition of Beta function in Eq. (1) and applying Eq. (4) we get Eq. (24) after integration when $Z_1 < n+1$, for some natural number n as

$$J_1 = \sum_{n=0}^{\infty} \frac{Z_2^n}{n!} B(1 + Z_1, 1 - Z_1 + n) = \Gamma(1 + Z_1) \sum_{n=0}^{\infty} \frac{Z_2^n}{n!} \frac{\Gamma(1 - Z_1 + n)}{\Gamma(2 + n)} \quad (25)$$

One can further write Eq. (25) applying Eq. (6) and (13) as

$$J_1 = \frac{\pi Z_1}{\sin \pi Z_1}\left[1+\sum_{n=1}^{\infty}\frac{Z_2^n}{n!}\frac{(1-Z_1)_n}{(2)_n}\right] \tag{26}$$

Similar type of equations (for $n = 0$ only) was mentioned by the authors in [3], [5] and [14]. The value of the integral J_1 can be calculated analytically using Eq. (26). In [6] author has shown that the secondary current has an effect on the vertical distribution of sediment concentration. Previously authors have neglected the effect of secondary current on sediment concentration distribution and the terms in the summation was not mentioned.

3.2 Integral J_2: Applying the approximation in Eq. (23), the integral J_2 can be written as

$$J_2 = \int_0^{\xi_a}\left(\frac{1-\xi}{\xi}\right)^{Z_1}(1+Z_2\xi)\,d\xi = L_0(Z_1)+L_1(Z_1) \tag{27}$$

in which

$$L_n(Z_1) = \int_0^{\xi_a}\left(\frac{1-\xi}{\xi}\right)^{Z_1} Z_2^n\xi^n d\xi \quad \text{for} \quad n=0,1 \tag{28}$$

The first integral in Eq. (27) can be expressed in terms of a fast converging series expansion according to [5] as

$$L_0(Z_1) = \xi_a\left(\frac{1-\xi_a}{\xi_a}\right)^{Z_1} - Z_1\sum_{k=1}^{\infty}\frac{(-1)^k}{k-Z_1}\left(\frac{\xi_a}{1-\xi_a}\right)^{k-Z_1} \tag{29}$$

The integral $L_1(Z_1)$ can be solved using integration by parts. It can be expressed as

$$L_1(Z_1) = -\frac{Z_2{\xi_a}^2}{2}\left(\frac{1-\xi_a}{\xi_a}\right)^{Z_1+1} + \frac{Z_2(1-Z_1)}{2}L_0(Z_1) \tag{30}$$

Inserting Eq. (29) and (30) into Eq. (27) and after simplification the integral J_2 can be expressed as

$$J_2 = \left[1-\frac{Z_2(Z_1-\xi_a)}{2}\right]\frac{(1-\xi_a)^{Z_1}}{{\xi_a}^{Z_1-1}} - Z_1\left[1+\frac{Z_2(1-Z_1)}{2}\right] \sum_{k=1}^{\infty}\frac{(-1)^k}{k-Z_1}\left(\frac{\xi_a}{1-\xi_a}\right)^{k-Z_1} \tag{31}$$

Therefore the integral I_1 can be expressed by inserting Eq. (26) and (31) into Eq. (21) as

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$$I_1 = \frac{\pi Z_1}{\sin \pi Z_1}\left[1+\sum_{n=1}^{\infty}\frac{Z_2^n}{n!}\frac{(1-Z_1)_n}{(2)_n}\right]-\left[1-\frac{Z_2(Z_1-\xi_a)}{2}\right]$$

$$\frac{(1-\xi_a)^{Z_1}}{\xi_a^{Z_1-1}}+Z_1\left[1+\frac{Z_2(1-Z_1)}{2}\right]\sum_{k=1}^{\infty}\frac{(-1)^k}{k-Z_1}\left(\frac{\xi_a}{1-\xi_a}\right)^{k-Z_1} \quad (32)$$

Equation (32) contains two infinite series corresponding to two integrals J_1 and J_2 respectively. The infinite series appears in J_2 is rapidly convergent because when $k-Z_1>1$, $\xi_a^{k-Z_1}$ rapidly tends to zero [5]. The convergence of the series in integral J_1 is relatively slower. Numerical investigations show that for $n=1$, Eq. (32) underestimates and for each value of $n=2,3,4...$ it overestimates from the actual numerical values for any value of the parameter Z_1 and values of parameter Z_2 satisfying $0 \le Z_2 \le 1$. An excellent approximation for $0 \le Z_2 < 0.1$ is obtained by taking the average of the two approximations for $n = 1$ and $n = 2$ as

$$I_1 = \frac{\pi Z_1}{\sin \pi Z_1}\left[1+\frac{Z_2}{2}(1-Z_1)+\frac{Z_2^{\,2}}{24}(1-Z_1)(2-Z_1)\right]$$

$$-\left[1-\frac{Z_2(Z_1-\xi_a)}{2}\right]\frac{(1-\xi_a)^{Z_1}}{\xi_a^{Z_1-1}}+Z_1\left[1+\frac{Z_2(1-Z_1)}{2}\right]\sum_{k=1}^{\infty}\frac{(-1)^k}{k-Z_1}\left(\frac{\xi_a}{1-\xi_a}\right)^{k-Z_1} \quad (33)$$

and for $0.1 \le Z_2 \le 1$ Eq. (32) is approximated as

$$I_1 = \frac{\pi Z_1}{\sin \pi Z_1}\left[1+\sum_{n=1}^{4}\frac{Z_2^n}{n!}\frac{(1-Z_1)_n}{(2)_n}+\frac{Z_2^{\,5}}{172800}\prod_{p=1}^{5}(p-Z_1)\right]$$

$$-\left[1-\frac{Z_2(Z_1-\xi_a)}{2}\right]\frac{(1-\xi_a)^{Z_1}}{\xi_a^{Z_1-1}}+Z_1\left[1+\frac{Z_2(1-Z_1)}{2}\right]\sum_{k=1}^{\infty}\frac{(-1)^k}{k-Z_1}\left(\frac{\xi_a}{1-\xi_a}\right)^{k-Z_1} \quad (34)$$

For larger values of the parameter $Z_2 (>1)$ one can take the average of the values of I_1 for n^{th} and $(n+1)^{\text{th}}$ (for some natural number n) terms to get the appropriate result. Equation (33) and (34) reverts to the formula proposed by [5] when $Z_2 = 0$. Computed values from Eq. (33) and (34) are plotted in Fig. 2 and Fig. 3 respectively for five different values of ξ_a and $0 \le Z_1 \le 1$ together with the numerical results. The value of the parameter Z_2 is indicated in the figure. From the figures one can conclude that the proposed approximations agree well with the numerical results. Calculated values of the integral I_1 from Eq. (34) for $Z_1 > 1$ are plotted in Fig. 4 with the numerical results for five different values of reference level ξ_a. Figure 4 shows the applicability of Eq. (34) clearly.

3.3 Integral J_3 : Using the Taylor series expansion of the exponential function, the integral J_3 can be written as

$$J_3 = \int_0^1 \left(\frac{1-\xi}{\xi}\right)^{Z_1} \ln\xi \left[\sum_{n=0}^{\infty} \frac{Z_2^n}{n!}\xi^n\right] d\xi \tag{35}$$

Before integrating Eq. (35), assume that

$$M_n = \frac{Z_2^n}{n!}\int_0^1 (1-\xi)^{Z_1}\,\xi^{n-Z_1}\ln\xi d\xi \quad \text{for } n = 0,1,2... \tag{36}$$

After using Eq. (6), (13) and (14), Eq. (36) can be written as

$$M_n = \frac{Z_2^n}{n!}\frac{\pi Z_1}{\sin\pi Z_1}\frac{(1-Z_1)_n}{(2)_n}\left[\psi(1-Z_1+n)-\psi(2+n)\right] \tag{37}$$

More explicitly one can simplify Eq. (37) as

$$M_n = D_n\left[\pi\cot\pi Z_1 - 1 - \frac{1}{Z_1} + \sum_{k=1}^{n}\left(\frac{1}{k-Z_1}-\frac{1}{k+1}\right) + \sum_{l=1}^{\infty}\left(\frac{1}{l}-\frac{1}{l+Z_1}\right)\right] \tag{38}$$

where

$$D_n = \frac{Z_2^n}{n!}\frac{\pi Z_1}{\sin\pi Z_1}\frac{(1-Z_1)_n}{(2)_n} \tag{39}$$

where Eq. (9), (10) and (12) has been applied. Therefore the integral J_3 can be expressed as

$$J_3 = \frac{\pi Z_1}{\sin\pi Z_1}\left[\pi\cot\pi Z_1 - 1 - \frac{1}{Z_1} + \sum_{l=1}^{\infty}\left(\frac{1}{l}-\frac{1}{l+Z_1}\right)\right] + \sum_{n=1}^{\infty} M_n \tag{40}$$

3.4 Integral J_4 : Similarly applying the approximation Eq. (23), the integral J_4 can be written as

$$J_4 = \int_0^{\xi_a}\left(\frac{1-\xi}{\xi}\right)^{Z_1}\ln\xi(1+Z_2\xi)\,d\xi = K_0(Z_1)+K_1(Z_1) \tag{41}$$

where

$$K_n(Z_1) = \int_0^{\xi_a}\left(\frac{1-\xi}{\xi}\right)^{Z_1}\ln\xi Z_2^n\xi^n d\xi \quad \text{for } n = 0,1 \tag{42}$$

According to [5] the first integral can be expressed in terms of a series expansion as

$$K_0(Z_1) = L_0(Z_1)\left(\ln\xi_a + \frac{1}{Z_1-1}\right) + Z_1\sum_{k=1}^{\infty}\frac{(-1)^k L_0(Z_1-k)}{(Z_1-k)(Z_1-k-1)} \tag{43}$$

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where $L_0(Z_1)$ is a function of Z_1 is given by Eq. (29). Integrating $K_1(Z_1)$ by parts and using Eq. (30) one can get

$$K_1(Z_1) = \frac{Z_2 \xi_a^{\ 2}}{2}\left(\frac{1-\xi_a}{\xi_a}\right)^{Z_1+1}\left\{\frac{1}{2} - \ln \xi_a\right\} + \frac{Z_2(1-Z_1)}{2} K_0(Z_1) + \frac{Z_2(1+Z_1)}{4} L_0(Z_1) \tag{44}$$

Therefore the integral J_4 can be expressed as

$$J_4 = \frac{Z_2 \xi_a^{\ 2}}{2}\left(\frac{1-\xi_a}{\xi_a}\right)^{Z_1+1}\left\{\frac{1}{2} - \ln \xi_a\right\} + \left[\left\{1+\frac{Z_2(1-Z_1)}{2}\right\}\left\{\ln \xi_a + \frac{1}{Z_1-1}\right\} + \frac{Z_2(1+Z_1)}{4}\right] L_0(Z_1) + Z_1\left\{1+\frac{Z_2(1-Z_1)}{2}\right\}\sum_{k=1}^{\infty}\frac{(-1)^k L_0(Z_1-k)}{(Z_1-k)(Z_1-k-1)} \tag{45}$$

and the integral I_2 can be expressed after inserting Eq. (40) and (45) into Eq. (22) as

$$I_2 = \frac{\pi Z_1}{\sin \pi Z_1}\left[\pi \cot \pi Z_1 - 1 - \frac{1}{Z_1} + \sum_{l=1}^{\infty}\left(\frac{1}{l} - \frac{1}{l+Z_1}\right)\right] + \sum_{n=1}^{\infty} M_n - \frac{Z_2 \xi_a^{\ 2}}{2}\left(\frac{1-\xi_a}{\xi_a}\right)^{Z_1+1}\left\{\frac{1}{2} - \ln \xi_a\right\} - \left[\left\{1+\frac{Z_2(1-Z_1)}{2}\right\}\left\{\ln \xi_a + \frac{1}{Z_1-1}\right\} + \frac{Z_2(1+Z_1)}{4}\right] L_0(Z_1) - Z_1\left\{\frac{2+Z_2(1-Z_1)}{2}\right\}\sum_{k=1}^{\infty}\frac{(-1)^k L_0(Z_1-k)}{(Z_1-k)(Z_1-k-1)} \tag{46}$$

Integration I_2 can be calculated analytically from Eq. (46). Equation (46) reverts to the formula proposed by [5] when parameter $Z_2 = 0$. Numerical calculations show that for $n = 1$, Eq. (46) agrees well with the numerical result for any value of parameter Z_1, and parameter Z_2 satisfying $0 \le Z_2 \le 1$. Therefore an appropriate approximation of Eq. (46) for all values of Z_1 can be taken by taking $n = 1$ in Eq. (46) as

$$I_2 = \frac{\pi Z_1}{\sin \pi Z_1}\left[1+\frac{Z_2(1-Z_1)}{2}\right]\left[\pi \cot \pi Z_1 - 1 - \frac{1}{Z_1} + \sum_{l=1}^{\infty}\left(\frac{1}{l} - \frac{1}{l+Z_1}\right)\right]$$
$$+\frac{\pi Z_1}{\sin \pi Z_1}\frac{Z_2(1+Z_1)}{4} - \frac{Z_2 \xi_a{}^2}{2}\left(\frac{1-\xi_a}{\xi_a}\right)^{Z_1+1}\left\{\frac{1}{2} - \ln \xi_a\right\}$$
$$-\left[\left\{1+\frac{Z_2(1-Z_1)}{2}\right\}\left\{\ln \xi_a + \frac{1}{Z_1 - 1}\right\} + \frac{Z_2(1+Z_1)}{4}\right] L_0(Z_1)$$
$$-Z_1\left\{1+\frac{Z_2(1-Z_1)}{2}\right\}\sum_{k=1}^{\infty}\frac{(-1)^k L_0(Z_1 - k)}{(Z_1 - k)(Z_1 - k - 1)} \qquad (47)$$

Comparisons between numerical result i.e., Eq. (20) with proposed analytical approximation Eq. (47) are plotted in Fig. 6 and Fig. 7 for $0 \le Z_1 \le 1$ and $1 \le Z_1 \le 5$ respectively for two different values of the parameter Z_2. Figure 6 and 7 show that the proposed analytical approximation of integral I_2 agrees well with the numerical results for all values of Rouse number Z_1 and parameter Z_2 satisfying $0 \le Z_2 \le 1$. Equations (34) and (47) contain three infinite series. Series in Eq. (29) and (43) are rapidly convergent and consideration of first ten terms in each of Eq. (29) and (43) gives accurate result [5]. The convergence of infinite series in Eq. (38) comparatively slow and an approximation proposed by [5] can be used in the program to make the calculation faster.

For any integer value of the Rouse number Z_1 the integral I_1 and I_2 cannot be calculated from Eq. (34) and Eq. (47) respectively as the function $\sin \pi Z_1$ approaches to zero when Z_1 approaches to any integer value. An integer Z_1 can be considered as $Z_1 = n \pm 10^{-3}$ [5]. When $Z_1 = n$ for some integer n, the solution in closed form can be obtained by using binomial theorem to the integrands I_1 and I_2 respectively as

$$I_1 = \int_{\xi_a}^{1}\left(\frac{1-\xi}{\xi}\right)^n \exp(Z_2 \xi)\, d\xi = \sum_{m=0}^{\infty}\frac{Z_2^m}{m!}\int_{\xi_a}^{1}(1-\xi)^n \xi^{m-n} d\xi$$
$$= \sum_{m=0}^{\infty}\frac{Z_2^m}{m!}\left[\sum_{k=0}^{n}\frac{(-1)^k n!}{k!(n-k)!}\int_{\xi_a}^{1}\xi^{k-n+m} d\xi\right]$$

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$$=\sum_{m=0}^{n-1\geq 0}\frac{Z_2^m}{m!}\left[\sum_{\substack{k=0\\(n-k-m)\neq 0,1}}^{n}\frac{(-1)^k\, n!}{k!(n-k)!}\frac{1-\xi_a{}^{k-n+1+m}}{(k-n+1+m)}\right]+$$

$$Z_2^n\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\frac{1-\xi_a{}^{k+1}}{k+1}+\sum_{m=n+1}^{\infty}\frac{Z_2^m}{m!}\left[\sum_{k=0}^{n}\frac{(-1)^k\, n!}{k!(n-k)!}\frac{1-\xi_a{}^{k-n+1+m}}{(k-n+1+m)}\right]$$

$$+(-1)^n\left(n\ln\xi_a+1-\xi_a\right)+\left(1-\xi_a\right)\sum_{l=1}^{n-1}(-1)^{n-l}\frac{Z_2^l}{l!}\binom{n}{l}+\ln\xi_a\sum_{l=1}^{n-1}(-1)^{n-l}\frac{Z_2^l}{l!}\binom{n}{l+1}\quad(48)$$

where $\binom{n}{k}$ denotes the binomial coefficient of indices n and k that occurs in binomial theorem. In particular when $n=0$ the exact solution of integral I_1 is given as $I_1=\dfrac{e^{Z_2}-e^{Z_2\xi_a}}{Z_2}$. It can be shown that by applying L'Hopital rule to this exact solution that it takes the form $I_1=1-\xi_a$ when $Z_2=0$,
and

$$I_2=\int_{\xi_a}^{1}\left(\frac{1-\xi}{\xi}\right)^n\ln\xi\exp\left(Z_2\xi\right)d\xi=\sum_{m=0}^{\infty}\frac{Z_2^m}{m!}\int_{\xi_a}^{1}(1-\xi)^n\,\xi^{m-n}\ln\xi d\xi$$

$$=\sum_{m=0}^{\infty}\frac{Z_2^m}{m!}\left[\sum_{k=0}^{n}\frac{(-1)^k\, n!}{k!(n-k)!}\int_{\xi_a}^{1}\xi^{k-n+m}\ln\xi d\xi\right]$$

$$=\sum_{m=0}^{n-1\geq 0}\frac{Z_2^m}{m!}\left[\sum_{\substack{k=0\\(n-k-m)\neq 0,1}}^{n}\frac{(-1)^k\, n!}{k!(n-k)!}\left\{\frac{\xi_a{}^{k-n+1+m}\ln\xi_a}{n-k-m-1}-\frac{1-\xi_a{}^{k-n+1+m}}{(n-k-m-1)^2}\right\}\right]$$

$$-Z_2^n\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\left\{\frac{\xi_a{}^{k+1}\ln\xi_a}{k+1}+\frac{1-\xi_a{}^{k+1}}{(k+1)^2}\right\}+$$

$$\sum_{m=n+1}^{\infty}\frac{Z_2^m}{m!}\left[\sum_{k=0}^{n}\frac{(-1)^k\, n!}{k!(n-k)!}\left\{\frac{\xi_a{}^{k-n+1+m}\ln\xi_a}{n-k-m-1}-\frac{1-\xi_a{}^{k-n+1+m}}{(n-k-m-1)^2}\right\}\right]$$

$$+(-1)^n\left\{\frac{n}{2}(\ln\xi_a)^2-\xi_a\ln\xi_a+\xi_a-1\right\}$$

$$+\frac{1}{2}(\ln\xi_a)^2\sum_{l=1}^{n-1}(-1)^{n-l}\frac{Z_2^l}{l!}\binom{n}{l+1}+(-\xi_a\ln\xi_a+\xi_a-1)\sum_{l=1}^{n-1}(-1)^{n-l}\frac{Z_2^l}{l!}\binom{n}{l}\quad(49)$$

To compute the integrals I_1 and I_2 one can use Eq. (33) or (34) and Eq. (47) respectively for any noninteger value of Rouse number Z_1 and Eq. (48) and Eq. (49) respectively for any integer value of Z_1. The value of the integrals I_1 and I_2

for any integer value of Z_1 is plotted in Fig. 8 and Fig. 9 respectively together with the numerical results. Figure 8 and 9 shows that the proposed approximation of these integrals for integral values of the Rouse number agrees well with the numerical results. Approximation Eq. (48) and (49) contains two infinite series which are rapidly convergent. In practice, taking $m = 10$ in these approximations will give accurate result.

4. TEST EXAMPLES:

4.1 Example 1: This example tests the proposed transport Eq. (18) with the experimental data collected at Huayuan Kuo Station, Yellow River, China [15, p.235]. To calculate the suspended-load transport rate from Eq. (18) the value of the integrals I_1 and I_2 should be known. The value of these integrals I_1 and I_2 is calculated from the proposed analytical approximations Eq. (33) or (34) and Eq. (47) respectively for noninteger values of Rouse number. The hydraulic conditions for this experiment were as follows: water depth $h = 2.7\ m$, temperature $T = 28.5^0 C$ and energy slope is $S = 7.2\times10^{-4}$. The field measurement data are shown in column 1-3 of Table 1. The suspended-load transport rate per unit width for the suspended regime is calculated through numerical integration of given data set and is equal to $q_s = 97.7331$ *kg/m/s* where the Rouse number $Z_1 = 0.159$ and $Z_2 = -0.013$ was obtained by fitting the data in column 3 of Table 1 with Eq. (17) using an iterative least square method. To compute suspended-load transport rate from Eq. (18) the reference level is taken as $\xi_a = 0.01$ and velocity and sediment concentration at reference level is calculated by extrapolating Eq. (16) and (17) respectively to the near bed region. The shear velocity is calculated from

$$u_* = \sqrt{ghS} = \sqrt{9.81\times2.7\times7.2\times10^{-4}} = 0.138\,\mathrm{m/s} \quad (50)$$

where $g = 9.81\ \mathrm{m/s^2}$ is the gravitational acceleration. Substitution of the value of the parameters into Eq. (34) and (47) gives $I_1 = 1.0128$ and $I_2 = -1.2072$ and suspended-load transport rate is given by Eq. (18) as

$$q_s = C_a u_* h\left(\frac{\xi_a}{1-\xi_a}\right)^{Z_1} \exp\left(-Z_2\xi_a\right)\left[\left(\frac{1}{\kappa}+\frac{U}{u_*}\right)I_1+\frac{1}{\kappa}I_2\right]$$

$$= 31.23\times0.138\times2.7\times\left(\frac{0.01}{1-0.01}\right)^{0.159}\times\exp\left\{-\left(-0.013\right)\times0.01\right\}\times$$

$$\left[\left(\frac{1}{0.4}+\frac{2.46}{0.138}\right)\times1.0128+\frac{1}{0.4}\left(-1.2072\right)\right]$$

$$= 98.2211\ \mathrm{kg/m/s} \quad (51)$$

which is very close to the value calculated from measurements. Using the direct numerical integration to the integrals I_1 and I_2, suspended-load transport rate is obtained from Eq. (18) as $q_s = 98.2216$ kg/m/s which slightly overestimates from the value obtained in Eq. (51). To get a quantitative idea of the error in the measurement the relative error is calculated from Eq. (52) as

$$\text{Relative error} = 1 - \frac{\text{approximate value}}{\text{numerical value}} \tag{52}$$

In both the cases the relative error is found to be less than 0.05%. From the above calculation one can observe that the calculated value of suspended-load transport rate using proposed analytical approximations is very close to the value calculated from measured data and also compares well with the numerical result. The relative error of the suspended-load transport rate in Eq. (51) compared to the numerical value is 0.0005%.

4.2 Example 2: This example test transport Eq. (18) for rough sands with the experimental data from Coleman's experiment [16] corresponds to the test case 36(coarse sand). Coleman [16] performed experiments with in a smooth flume of 356 mm width and 15 m long. The height and energy slope of this experiment was 0.171 m and 0.002 respectively. The shear velocity was 0.041 m/s. The reference velocity and concentration was calculated by extrapolating the corresponding log law and Rouse equation after fitting with the data. The value of Rouse number is 1.28. Substituting the data in Eq. (34) and (47) one can get $I_1 = 12.0213$ and $I_2 = -28.0147$ and finally the suspended load transport rate is calculated from Eq. (18) using the analytical approximations as $q_s = 0.007478$ kg/m/s. Using the direct numerical evaluation the suspended-load transport rate is $q_s = 0.007443$ kg/m/s. The relative error is less than 1%.

Table 1 Yellow river data for Calculation of suspended-load transport rate (* denotes extrapolated values)

Relative distance from bed $\xi(= y/h)$	Measured velocity u (m/s)	Measured concentration C (kg/m^3)
0.01 (ξ_a)	1.22*	31.23*
0.04	1.64	24.90
0.10	1.86	20.30
0.20	2.10	19.80
0.30	2.31	17.10
0.40	2.46	16.00
0.60	2.74	14.00
0.80	2.96	12.20
0.96	3.11	8.140
1.00	-	0.000

5. REFERENCES

1. H. A. Einstein, “The Bed Load Function for Sediment Transport in Open Channel Flows,” Technical Bulletin no. 1026, USDA Soil Conservation Services, Washington, D.C, 1950.
2. T. Nakato, “Numerical integration of Einstein’s Integrals, I_1 and I_2,” J. Hydr. Engrg., ASCE, 110(12), 1984, pp. 1863-1868.
3. J. Guo, and W. L. Wood, “Fine Suspended sediment transport rates,” J. Hydr. Engrg., ASCE, 121(12), 1995, pp. 919-922.
4. J. Guo, “Approximations of Gamma function and Psi function and their applications in Sediment Transport,” Proc. 13th Congress on Advances in Hydraulics and Water Engineering by IAHR-APD, Singapore, 2002, pp. 219-223.
5. J. Guo, and P. Y. Julien, “Efficient Algorithm for Computing Einstein Integrals,” J. Hydr. Engrg., ASCE, 130(12), 2004, pp. 1198-1201.
6. S. Q. Yang, “Turbulent transfer mechanism in sediment-laden flow,” J. Geo. Res., 112, 2007, pp. 1-14.
7. S. Kundu, and K. Ghoshal, “Vertical distribution of sediment concentration in open channels from two-phase flow analysis,” unpublished.
8. W. Rudin, Principles of Mathematical Analysis, McGraw-Hill Inc, 1976.
9. P. Sebah, and X. Gourdon, Introduction to the Gamma Function, http://www.profesores.frc.utn.edu.ar/electronica/analisisdeseniales/aplicaciones/Funcion_Gamma.pdf, 2002.
10. M. Abramowitz, and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publication, New York, 1972.
11. N. Chien, and Z. H. Wan, Sediment Transport Mechanics, ASCE Press, Reston, VA, 1999.
12. P. Y. Julien, Erosion and Sedimentation, Cambridge University Press, Cambridge, 1955.
13. W. H. Graf, Hydraulics of Sediment Transport, Water Resources Publication, Littleton, CO, USA, 1984.
14. J. Guo, and Y. Hui, “A further study on Einstein’s sediment transport theory,” Advances in Water Science, Water Resources Press, Beijing, 2(2), 1991, pp. 81-91.
15. J. Guo, “Turbulent Velocity Profiles in Clear Water and Sediment-laden Flows,” Ph. D. thesis, Colorado State University, Fort Collins, Colorado, 1998.
16. N. L. Coleman, “Effects of Suspended Sediment on the Open Channel Velocity Distribution”, Water Resources Res., 22(10), 1986, pp. 1377-1384.

FIGURES

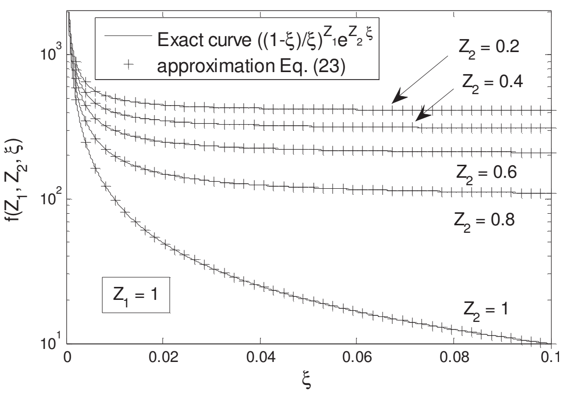

Figure 1 Comparison between exact and approximated values of $f(Z_1, Z_2, \xi)$.

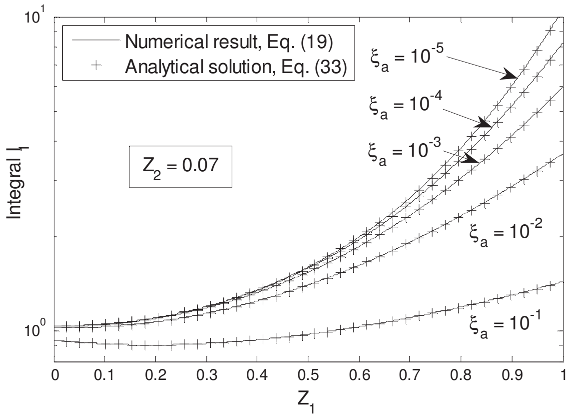

Figure 2 Comparison of results of integral I_1 for $Z_2 < 0.1$.

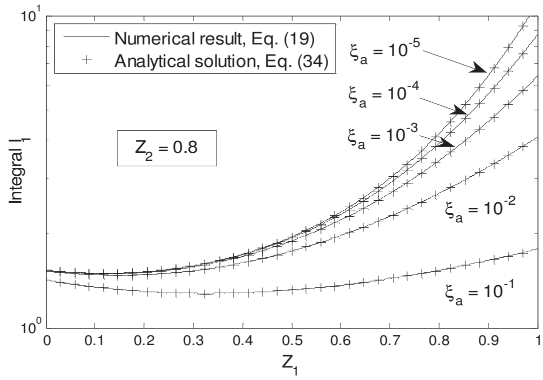

Figure 3 Comparison of results of integral I_1 for $Z_2 \geq 0.1$.

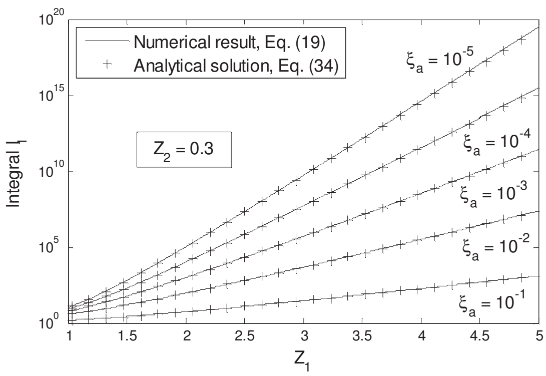

Figure 4 Plot of Integral I_1.

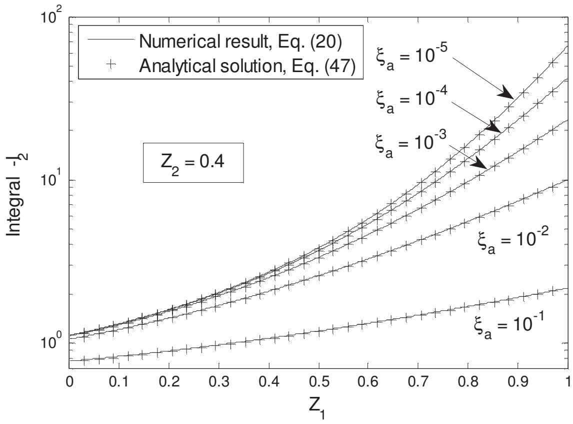

Figure 5 Comparison of results of integral I_2 for $0 \le Z_1 \le 1$ (negative y -axis).

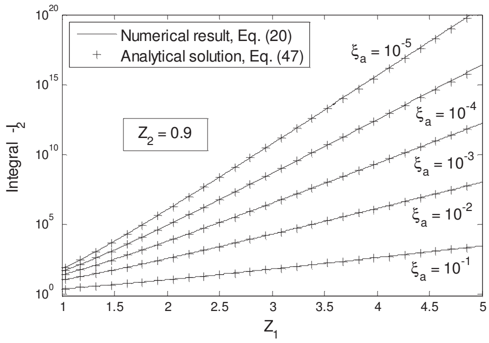

Figure 6 Plot of Integral I_2 (negative y -axis).

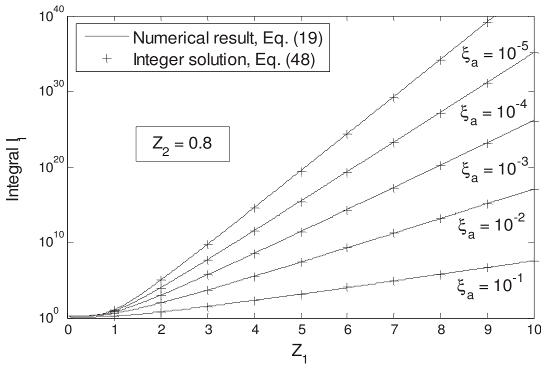

Figure 7 Plot of Integral I_1 for integer values of Z_1.

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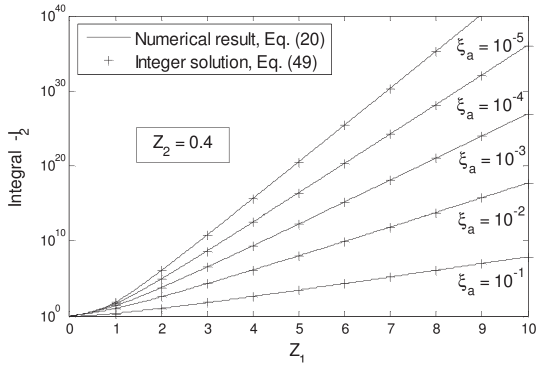

Figure 8 Plot of Integral I_2 for integer values of Z_1 .

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