THE 4x n MULTISTATE LIGHTS OUT GAME

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Abstract: In this paper we explore the multistate Lights Out game, in which 4 x n rectangular Lights Out games have buttons that can take on more than two states. A linear algebra approach will be discussed and several classes of games will be explored.

Keywords: Lights out, applied linear algebra.

The original Tiger Electronics hand-held Lights Out game offers the user a 5x5 grid of on and off lights and challenges them to make a sequence of presses to turn all the lights out. Stemming from this game, a variety of alternative popular Lights Out games have been well studied. These include two state Lights Out game which have been studied in detail, exploring grids of different sizes, initial states of lights, and connections between lights [1], [2], [4], [6], [7], [9]. In addition, Lights Out games starting with lights that are all on and focusing on getting all of the lights off have been analyzed in full [3].

The traditional Lights Out games are binary games with analyses that include a variety of parity arguments. This paper focuses on solving the Lights Out game with buttons which can take on multiple states, called the **Multistate Lights Out Game** or **multistate game**, on a $4 \times n$ grid.

The Lights Out Grid: In a Lights Out grid, connections can be represented with an adjacency matrix, where a button is considered adjacent to itself since when it is pressed it both toggles itself and adjacent buttons to the next state. For example, in the 4×3 Lights Out grid, connections can be represented by a 12×12 adjacency matrix, M.

1	2	3
4	5	6
7	8	9
10	11	12

Figure 1: 4x3 Lights Out Grid

$$M = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}$$

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In general, the m × n Lights Out grid can be represented by a block tridiagonal matrix similar to

$$M = \begin{pmatrix} B & I & 0 \\ I & B & I \\ 0 & I & B \end{pmatrix}$$

M consists of *m* block rows and columns with *I* representing the n×n identity matrix, 0 the nxn zero matrix, and *B* the tridiagonal n×n matrix where $B(i,j) = \begin{cases} 1 & if\ i = j, j+1, j-1, \\ 0 & otherwise. \end{cases}$

One can also represent the sequence of pushes as a vector, $\vec{p}$, where $M \cdot \vec{p}$ represents the buttons toggled under the presses in $\vec{p}$. This paper explores the question in the multistate game of showing that there is a solution which changes buttons which are all in the same state, state 0, to buttons in another state, state 1, which is equivalent to determining if there is a solution to $M \cdot \vec{p} = \vec{1}$.

Characteristics of both *M* and *B* must be explored further in order to find a solution to this problem. Let M_m represents the m × m block matrix, *M*, and B_n to represent the n × n tridiagonal matrix, *B*.

Lemma 1. $|M_m| = |B \cdot |M_{m-1}| - |M_{m-2}||$

Lemma 2. $|B_n| = |B_{n-1}| - |B_{n-2}|$

Solutions to the 4 × *n* multistate game: One can quickly see that the 1 × *n* and 2 × *n* multistate Lights Out games are solvable for any number of button states, n. Here we turn our attention to the next obvious choice of Lights Out games, those of size 4 × *n*. We begin with interesting patterns that can be seen when $m \equiv 4(mod5)$.
Note that $|M_4| = |B^2 - B - I||B^2 + B - I|$.

Lemma 3. $|M_m|$ contains factors $|B^2 - B - I|$ and $|B^2 + B - I|$ for m ≡ 4(mod5).

Proof. Assume that for m = 4 + 5k, $|M_m|$ contains factors $|B^2 - B - I|$ and $|B^2 + B - I|$,for k ∈ **Z**. Inductively, using Lemma 1, it can be shown that
$|M_{m+5}| = |(B^2 - B - I)(B^2 + B - I) \cdot |M_{m+1}| + (B - I) \cdot |M_m||$.

To say more about the 4×n games, the behavior of $|B^2 - B - I|$ and $|B^2 + B - I|$, must be explored further. First note that in Z_p,

$$|B_n^2 - B_n - I| = \begin{cases} p-1, & if\ n \equiv 1,2\ (mod\ 5), \\ 1, & if\ n \equiv 0,3\ (mod\ 5), \\ 0, & if\ n \equiv 4\ (mod\ 5). \end{cases}$$

Theorem 1. *The matrix M affiliated with the m× n multistate game with m,n* ≡ 4(*mod*5) *and p states is singular.*

Theorem 2. *The* 4 × *n multistate game with* $n \not\equiv 3(mod\ 4)$ *and* $n \not\equiv 4(mod\ 5)$ *and p states has a solution, where* $p = 3^k$, $k \in \mathbf{Z}$.

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Proof: With a small amount of work one can see that $|B_n^2 - B_n - I| \not\equiv 0(mod\ 3^k)$ for $n \not\equiv 3(mod\ 4)$ and $n \not\equiv 4(mod\ 5)$, $k \in \mathbf{Z}$. Denote $T_n = (B_n^2 - B_n - I)(B_n^2 + B_n - I)$ and

$$T_n(i,j) = \begin{cases} 1, & i = j+2, j = i+2, \\ 2, & i = j = 1, i = j = n, \\ 3, & i = j+1, j = i+1, i = j \neq 1, n, \\ 0, & otherwise. \end{cases}$$

Assume that $n \not\equiv 3(mod\ 4)$ and $n \not\equiv 4(mod\ 5)$, we will show that $|T_n| \not\equiv 0(mod\ 3^k)$. Note that since $n \not\equiv 3(mod\ 4)$ neither is n-4. Assume that $|T_{n-4}| \not\equiv 0(mod\ 3^k)$ and thus T_{n-4} is nonsingular in $\mathbf{Z}^{3k}$. T_n can be expressed as the block matrix $T_n = \begin{pmatrix} T_{n-4} & A_1 \\ A_2 & T_4 \end{pmatrix}$ and
$|T_n| = |T_{n-4}||T_4 - A_2 \cdot (T_{n-4})^{-1} \cdot A_1|$.

$(T_4 - A_2 \cdot (T_{n-4})^{-1} \cdot A_1)$ takes on one of only three 4×4 matrices, dependent of the value of $n(mod 4)$ and k. Each of the three values of

$|T_4 - A_2 \cdot (T_{n-4})^{-1} \cdot A_1| \not\equiv 0(mod\ 3^k)$ for all $k \in \mathbf{Z}$ and
thus $|T_n| \not\equiv 0(mod\ 3^k)$ for all $k \in \mathbf{Z}$. Thus,

$$|M_4| = |B^2 - B - I||B^2 + B - I| \not\equiv 0(mod 3^k)$$

and we can conclude that $4 \times n$ multistate game with $n \not\equiv 3(mod\ 4)$, $n \not\equiv 4(mod\ 5)$,and 3^k states, for $k \in \mathbf{Z}$ has a solution.

Example 1. The 4 × 12 multistate Lights Out game with 9 states has a solution. The numbers in the grids presented in Figure 2 represent the number of presses needed on that button to solve the game. The 4 × 12 grid on the left in Figure 2, shows the presses taking all lights in state 0 to state 1 (if these presses are done repetitively one can go from state 0 to any other state). Due to the fact that there are an odd number, 9, of states in this game, we can also solve this game using the presses shown n right 4 × 12 grid in Figure 2. These presses take all buttons in state 0 to state 2, and again with repetition one can go from all state 0 to all lights in any other state.

3	5	8	0	6	1	1	6	0	8	5	3
2	3	6	5	3	2	2	3	5	6	3	2
2	3	6	5	3	2	2	3	5	6	3	2
3	5	8	0	6	1	1	6	0	8	5	3

6	1	7	0	3	2	2	3	0	7	1	6
4	6	3	1	6	4	4	6	1	3	6	4
4	6	3	1	6	4	4	6	1	3	6	4
6	1	7	0	3	2	2	3	0	7	1	6

Figure 2. Solutions to the 4 × 12 game with 9 states

With this exploration of 4 × *n* multistate Lights Out games, one might consider other multistate Lights Out games with solutions. A more cumbersome look at multistate Lights Out grids of this size with infinitely many solutions could be further explored. In addition, a further exploration of *m*×*n* games multistate games with $m \equiv 4(mod5)$ may produce interesting results.

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