OPTIMAL HARVESTING OF A THREE SPECIES ECOLOGICAL MODEL WITH A PREY, PREDATOR AND A COMPETITOR

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***Abstract:** The present paper is devoted to an analytical investigation of three species ecological model with a Prey (N_1), a predator (N_2) and a competitor (N_3) . The third species (N_3) competes with the to the Prey (N_1) and the Predator (N_2) . Further, the prey and predator are harvested . In addition to that, all three species are provided with alternative food. The model is characterized by a set of first order non-linear ordinary differential equations. Equilibrium points of the model are identified and the local stability is discussed using Routh-Hurwitz certiria, global stability by Lyapunov's function. The existence of bionomic equilibrium of the system has been discussed and optimal harvesting policy is given using Pontryagin's maximum principle .The analytical discussion is supported by Numerical simulation using Mat lab.*

***Keywords:** Prey, Predator, competitor, Equilibrium points, Bionomic Equilibrium, Optimal harvesting policy, Pontryagin's maximum principle, Numerical simulation.*

1. INTRODUCTION:

Ecology relates to the study of living beings (animals and plants) in relation to their habits and habitats. This discipline of knowledge is a branch of evolutionary biology purported to explain how or what extent the living beings are regulated in nature. Research in the area of theoretical ecology was initiated in 1925 by Lotka [5] and by Volterra [6]. The general concept of modelling have been presented in the treatises of Paul Colinvaux [4], Freedman [1], Kapur [2, 3] etc. The ecological symbiosis can be broadly classified recently as prey, predation, competition, mutualism, commensalism, amensalism and so on. Recently Shiva Reddy [7] discussed the stability analysis of three species eco system consting of prey, predator and super predator and Paparao [8] discussed a prey, predator model with a cover linearly varying with the prey population and an alternative food for the predator.

Bio-economic modeling of the exploitation of biological recourses such as fisheries and forests gained importance in resent years .the techniques and issues associated with bionomic and biological of renewable recourses have been studied by Clark C.W [12], optimal harvesting of renewable recourses have been extensively studied by T.K Kar[10], Chaudhuri..K.S [9], Shiva reddy.K [11], Ragozin.D.L and Brown.G [13] and Dubey.B [14]. Inspired from that, we discussed a more general problem of harvesting two species (prey& predator) of three species Ecological model, with a prey, predator and competitor to both the prey and predator, Further, the prey and predator are harvested optimally. The model is characterized by a set of first order ordinary differential equations. The local and global stability analyses of the system have been carried out. The bionomic equilibrium and optimal harvesting policy are derived using pontryagin's maximal principle and results are supported by numerical simulations.

2. BASIC EQUATIONS

The model equations for a three species Prey - Predator and competitor to the prey and predator system is given by the following system of first order ordinary differential equations employing the following notation:

$$\frac{dN_1}{dt} = a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1N_2 - \alpha_{13}N_1N_3 - q_1E_1N_1$$
$$\frac{dN_2}{dt} = a_2N_2 - \alpha_{22}N^2 + \alpha_{21}N_1N_2 - \alpha_{23}N_3N_2 - q_2E_2N_2 \qquad (2.1)$$
$$\frac{dN_3}{dt} = a_3N_3 - \alpha_{33}N_3^2 - \alpha_{31}N_1N_3 - \alpha_{32}N_2N_3$$

Where N_1 , N_2 and N_3 are the populations of the prey and predator and a competitor to the prey and predator with the natural growth rates a_1 a_2 and a_3 respectively,

α_{11} is rate of decrease of the prey due to insufficient food and inter species competition,

α_{12} is rate of decrease of the prey due to inhibition by the predator,

α_{21} is rate of increase of the predator due to successful attacks on the prey,

α_{22} is rate of decrease of the predator due to insufficient food other than the prey and inter species competition,

α_{23} is rate of decrease of the predator due to the competition with the third species

α_{33} is rate of decrease of the Competitor to the both prey and predator due to insufficient food and inter species competition,

α_{32} is rate of decrease of the competitor due to the competition with the both prey and predator,

α_{13} is rate of decrease of the prey due to the competition with the both prey and predator,

α_{31} is rate of decrease of the competitor due to the competition with the both prey and predator.

q_1 : Catch ability coefficient of prey (N_1), q_2 : catch ability coefficient of predator (N_2),

E_1: Effort applied to harvest the prey (N_1), E_2: Effort applied to harvest the predator (N_2)

Through out the analysis assume $(a_1 - q_1 E_1) > 0$ and $(a_2 - q_2 E_2) > 0$

3. EQUILIBRIUM STATES:

The system under investigation has eight equilibrium states. They are

I.E_1: The extinct state $\overline{N_1} = 0; \overline{N_2} = 0, \overline{N_3} = 0$ (3.1)

II.E_2: The state in which only the predator survives and the prey and competitor to the prey and predator are extinct

$$\overline{N_1} = 0, \overline{N_2} = \frac{(a_2 - q_2 E_2)}{\alpha_{22}}, \overline{N_3} = 0 \quad (3.2)$$

III. E_3: The state in which both the prey and the predators extinct and competitor to the prey and predator survive

$$\overline{N_1} = 0; \overline{N_2} = 0 \quad \overline{N_3} = \frac{a_3}{\alpha_{33}} \quad (3.3)$$

IV. E_4: The state in which both the predator and competitor to the prey and predator extinct and prey alone survive

$$\overline{N_1} = \frac{(a_1 - q_1 E_1)}{\alpha_{11}}; \quad \overline{N_2} = 0; \quad \overline{N_3} = 0 \quad (3.4)$$

V. E_5 :The state in which both the prey and the predators exist and competitor to the prey and predator extinct

$$\overline{N_1} = \frac{((a_1 - q_1 E_1)\alpha_{22} - (a_2 - q_2 E_2)\alpha_{12})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}};$$

$$\overline{N_2} = \frac{((a_2 - q_2 E_2)\alpha_{11} + (a_1 - q_1 E_1)\alpha_{21})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}; \quad \overline{N_3} = 0$$

This case arise only when $(a_1 - q_1 E_1)\alpha_{22} > (a_2 - q_2 E_2)\alpha_{12}$ (3.5)

VI. E_6 :The state in which both prey and competitor to the prey and predator exist and predator extinct

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$$\overline{N_1} = \frac{((a_1 - q_1E_1)\alpha_{33} - a_3\alpha_{13})}{(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31})} \qquad \overline{N_2} = 0;$$

$$\overline{N_3} = \frac{(a_3\alpha_{11} - (a_1 - q_1E_1)\alpha_{31})}{(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31})} \tag{3.6}$$

The equilibrium state exist only when $\alpha_{11}\alpha_{33} > \alpha_{13}\alpha_{31}$, $(a_1 - q_1E_1)\alpha_{33} > a_3\alpha_{13}$

And $a_3\alpha_{11} > (a_1 - q_1E_1)\alpha_{31}$ (3.7)

VII. E_7: The state in which both predator and competitor to the prey and predator exist and prey extinct

$$\overline{N_1} = 0; \overline{N_2} = \left(\frac{(a_2 - q_2E_2)\alpha_{33} - a_3\alpha_{23}}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}}\right);$$

$$\overline{N_3} = \left(\frac{a_3\alpha_{22} - (a_2 - q_2E_2)\alpha_{32}}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}}\right) \tag{3.8}$$

The equilibrium state exist only when $(a_2 - q_2E_2)\alpha_{33} - a_3\alpha_{23}$, $a_3\alpha_{22} > (a_2 - q_2E_2)\alpha_{32}$

And $\alpha_{22}\alpha_{33} > \alpha_{23}\alpha_{32}$

VIII. E_8 :The state in which prey, predator and competitor to the prey and predator exist

$$\overline{N_1} = \frac{(a_1 - q_1E_1)(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) - \alpha_{12}((a_2 - q_2E_2)\alpha_{33} - a_3\alpha_{23}) + \alpha_{13}((a_2 - q_2E_2)\alpha_{32} - a_3\alpha_{22})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22})}$$

$$\overline{N_2} = \frac{((a_2 - q_2E_2)\alpha_{11}\alpha_{33} + (a_1 - q_1E_1)\alpha_{21}\alpha_{33} + (a_1 - q_1E_1)\alpha_{31}\alpha_{23}) - (a_3\alpha_{11}\alpha_{23} + a_3\alpha_{21}\alpha_{13} + (a_2 - q_2E_2)\alpha_{13}\alpha_{31})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22})}$$

$$\overline{N_3} = \frac{(a_3\alpha_{11}\alpha_{22} + a_3\alpha_{21}\alpha_{12} + (a_2 - q_2E_2)\alpha_{12}\alpha_{31}) - ((a_2 - q_2E_2)\alpha_{11}\alpha_{32} + (a_1 - q_1E_1)\alpha_{21}\alpha_{32} + (a_1 - q_1E_1)\alpha_{31}\alpha_{22})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22})} \tag{3.9}$$

The equilibrium state exist only when $\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}(\alpha_{21}\alpha_{33}+\alpha_{31}\alpha_{23})>\alpha_{13}(\alpha_{21}\alpha_{32}+\alpha_{31}\alpha_{22})$,

$$(a_1-q_1E_1)(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{13}((a_2-q_2E_2)\alpha_{32}-a_3\alpha_{22})>\alpha_{12}((a_2-q_2E_2)\alpha_{33}-a_3\alpha_{23})$$
,

$$((a_2-q_2E_2)\alpha_{11}\alpha_{33}+(a_1-q_1E_1)\alpha_{21}\alpha_{33}+(a_1-q_1E_1)\alpha_{31}\alpha_{23})>(a_3\alpha_{11}\alpha_{23}+a_3\alpha_{21}\alpha_{13}+(a_2-q_2E_2)\alpha_{13}\alpha_{31})$$

And

$$(a_3\alpha_{11}\alpha_{22}+a_3\alpha_{21}\alpha_{12}+(a_2-q_2E_2)\alpha_{12}\alpha_{31})>((a_2-q_2E_2)\alpha_{11}\alpha_{32}+(a_1-q_1E_1)\alpha_{21}\alpha_{32}+(a_1-$$
3.10)

4 STABILITY OF THE EQUILIBRIUM STATES:

Let $N=(N_1,N_2,N_3)^T=\overline{N}+U$ (4.1)

Where $U=(u_1,u_2,u_3)^T$ is the perturbation over the equilibrium state. $\overline{N}=(\overline{N_1},\overline{N_2},\overline{N_3})^T$ .The basic equations (2.1) are linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt}=AU \quad (4.2)$$

Where

$$A=\begin{bmatrix} a_1-2\alpha_{11}N_1-\alpha_{12}N_2-\alpha_{13}N_3-q_1E_1 & -\alpha_{12}N_1 & -\alpha_{13}N_1 \\ \alpha_{21}N_2 & a_2-2\alpha_{22}N_2+\alpha_{21}N_1-\alpha_{23}N_3-q_2E_2 & -\alpha_{23}N_2 \\ -\alpha_{31}N_1 & -\alpha_{32}N_3 & a_3-2\alpha_{33}N_3-\alpha_{31}N_1-\alpha_{32}N_2 \end{bmatrix}$$

The characteristic equation for the system is $det\left[A-\lambda I\right]=0$ (4.3)

The equilibrium state is stable, if three roots of the equation (4.3) are negative in case they are real or the roots have negative real parts in case they are complex.

4.1. Stability of the equilibrium state E_8:

The variational matrix for the interior equilibrium state is

$$A = \begin{bmatrix} -\alpha_{11}N_1 & -\alpha_{12}N_{11} & -\alpha_{13}N_1 \\ \alpha_{21}N_2 & -\alpha_{22}N_2 & -\alpha_{23}N_2 \\ -\alpha_{31}N_1 & -\alpha_{32}N_3 & -\alpha_{33}N_3 \end{bmatrix} \tag{4.1.1}$$

the characteristic equation of interior equilibrium state is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{4.1.2}$$

where

$$b_1 = \alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2 + \alpha_{33}\overline{N}_3$$

$$b_2 = (\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31})\overline{N_1}\,\overline{N_3} + (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\overline{N_1}\,\overline{N_2} + (\alpha_{33}\alpha_{22} - \alpha_{23}\alpha_{32})\overline{N_2}\,\overline{N_3}$$

and

$$b_3 = (\alpha_{11}\alpha_{33}\alpha_{22} + \alpha_{21}\alpha_{12}\alpha_{33} + \alpha_{23}\alpha_{12}\alpha_{31}) - (\alpha_{13}\alpha_{31}\alpha_{22} + \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{21}\alpha_{32}\alpha_{13})\overline{N_1}\,\overline{N_2}\,\overline{N_3}$$

By Routh-Hurwitz criteria, when all Eigen values of the above characteristic equation have negative real parts if only if $b_1 > 0$, $(b_1b_2 - b_3) > 0$ and $b_3(b_1b_2 - b_3) > 0$ .clearly $b_1 > 0$ and $b_3 > 0$ if $\left[(\alpha_{11}\alpha_{33}\alpha_{22} + \alpha_{21}\alpha_{12}\alpha_{33} + \alpha_{23}\alpha_{12}\alpha_{31}) > (\alpha_{13}\alpha_{31}\alpha_{22} + \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{21}\alpha_{32}\alpha_{13})\right]$
And based on certain algebraic deductions applicable in this case, it can be verified that $b_3(b_1b_2 - b_3) > 0$.

Therefore the roots of (4.1.1) are real and negative or complex conjugates having negative real parts.

Thus the system is locally stable for interior equilibrium point $E_8\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$

5. GLOBAL STABILITY:

Theorem: The Equilibrium point $E_8(\overline{N}_1, \overline{N}_2, \overline{N}_3)$ is globally asymptotically stable. If $\alpha_{12} \geq \alpha_{21}$

Proof: Let us consider the following Lyapunov function

$$V(\overline{N}_1,\overline{N}_2,\overline{N}_3) = \left\{N_1 - \overline{N}_1 - \overline{N}_1 \ln\left[\frac{N_1}{\overline{N}_1}\right]\right\} + \left\{N_2 - \overline{N}_2 - \overline{N}_2 \ln\left[\frac{N_2}{\overline{N}_2}\right]\right\} + \left\{N_3 - \overline{N}_3 - \overline{N}_3 \ln\left[\frac{N_3}{\overline{N}_3}\right]\right\}$$

(5.1)

Differentiating V w.r.to 't' we get

$$\frac{dV}{dt}=\left(\frac{N_1-\overline{N}_1}{N_1}\right)\frac{dN_1}{dt}+\left(\frac{N_2-\overline{N}_2}{N_2}\right)\frac{dN_2}{dt}+\left(\frac{N_3-\overline{N}_3}{N_3}\right)\frac{dN_3}{dt} \quad (5.2)$$

$$\frac{dV}{dt}=-\left(\alpha_{11}+\frac{(\alpha_{12}-\alpha_{21}+\alpha_{13}+\alpha_{31})}{2}\right)\left[N_1-\overline{N}_1\right]^2-\left(\alpha_{22}+\frac{(\alpha_{12}-\alpha_{21}+\alpha_{13}+\alpha_{31})}{2}\right)\left[N_2-\overline{N}_2\right]^2$$
$$-\left(\alpha_{33}+\frac{(\alpha_{12}-\alpha_{21}+\alpha_{13}+\alpha_{31})}{2}\right)\left[N_3-\overline{N}_3\right]^2 \quad (5.3)$$

$\frac{dV}{dt}<0$ If $\alpha_{12}\geq\alpha_{21}$

Therefore , $E_8(\overline{N}_1,\overline{N}_2,\overline{N}_3)$ is globally asymptotically stable

6. BIO ECONOMIC ASPECT AT INTERIOR EQUILIBRIUM POINT:

The term bionomic equilibrium is a combination of the biological equilibrium and economic equilibrium .The biological equilibrium is given by Equation (2.1) and economic equilibrium is said to be achieved when TR(The total revenue obtained selling the harvested biomass) equals to TC(The total cost for the effort devoted to the harvest) .It is the study of the dynamics of living resources using mathematical models that are drawn from simple economic models. As mentioned earlier, a biological equilibrium is given by $\frac{dN_i}{dt}=0$, i=1, 2, 3. The bionomic equilibrium is said to be achieved when the selling price of the harvested biomass equals to the total cost price utilized in harvesting it. Let c_1 be the harvesting cost per unit effort for prey species, c_2 be the harvesting cost per unit effort for predator species, p_1 be the price per unit biomass of the prey, p_2 be the price per unit biomass of the predator. Therefore net revenue or economic rent at any time given by $R=R_1+R_2$. Where $R_1=(p_1q_1N_1-c_1)E_1$, $R_2=(p_2q_2N_3-c_2)E_2$

here R_1 and R_2 represent net revenue for the prey, the predator respectively.

The bionomic equilibrium $\left((N_1)_\infty,(N_2)_\infty,(N_3)_\infty,\left(E_1\right)_\infty,\left(E_2\right)_\infty\right)$ is given by the following equations.

$$a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1N_2 - \alpha_{13}N_1N_3 - q_1E_1N_1 = 0 \tag{6.1}$$

$$a_2N_2 + \alpha_{21}N_2N_1 - \alpha_{22}N_2^2 - \alpha_{23}N_2N_3 - q_2E_2N_2 = 0 \tag{6.2}$$

$$a_3N_3 - \alpha_{31}N_1N_3 - \alpha_{32}N_2N_3 - \alpha_{33}N_3^2 = 0 \tag{6.3}$$

The revenue returns (R) on prey and predators is given by

$$R = \left(p_1q_1N_1 - c_1\right)E_1 + \left(p_2q_2N_3 - c_2\right)E_2 \tag{6.4}$$

In order to determine the bionomic equilibrium we come across the following cases.

Case (i): if $c_2 > p_2q_2N_3$ then the cost is greater than revenue for predator then the predator species will be stopped (E_2=0). Only the prey species remains operational. $\left(c_1 < p_1q_1N_1\right)$

$$\left(N_1\right)_\infty = \frac{c_1}{p_1q_1} \tag{6.5}$$

$$\left(N_2\right)_\infty = \frac{1}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}}\left(a_2\alpha_{33} + (\alpha_{21}\alpha_{33} + \alpha_{23}\alpha_{31})\frac{c_1}{p_1q_1} - a_3\alpha_{23}\right) \tag{6.6}$$

$$\left(N_3\right)_\infty = \frac{1}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}}\left(a_3\alpha_{22} - a_2\alpha_{32} - (\alpha_{31}\alpha_{22} + \alpha_{32}\alpha_{21})\frac{c_1}{p_1q_1}\right) \tag{6.7}$$

$$\left(E_1\right)_\infty = \frac{1}{q_1}\left\{a_1 - \alpha_{11}\frac{c_1}{p_1q_1} - \alpha_{12}\left(N_2\right)_\infty - -\alpha_{13}\left(N_3\right)_\infty\right\} \tag{6.8}$$

Now $\left(E_1\right)_\infty > 0$, if the following condition holds

$$a_1 > \left(\alpha_{11}\frac{c_1}{p_1q_1} + \alpha_{12}\left(N_2\right)_\infty + \alpha_{13}\left(N_3\right)_\infty\right) \tag{6.9}$$

Case (ii): If $c_1 > p_1q_1N_1$ then the cost is greater than revenue for prey then the prey species will be stopped (E_1=0). Only the predator species remains operational $c_2 < p_2q_2N_3$

$$\left(N_2\right)_\infty = \frac{c_2}{p_2 q_2} \tag{6.10}$$

$$\left(N_1\right)_\infty = \frac{1}{\alpha_{11}\alpha_{33} - \alpha_{31}\alpha_{13}}\left(a_1\alpha_{33} + (\alpha_{13}\alpha_{32} - \alpha_{12}\alpha_{33})\frac{c_2}{p_2 q_2} - a_3\alpha_{13}\right) \tag{6.11}$$

$$\left(N_3\right)_\infty = \frac{1}{\alpha_{11}\alpha_{33} - \alpha_{31}\alpha_{13}}\left(a_3\alpha_{11} + (\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})\frac{c_2}{p_2 q_2} - a_1\alpha_{31}\right) \tag{6.12}$$

Now substituting $\left(N_1\right)_\infty, \left(N_2\right)_\infty, \left(N_3\right)_\infty$ in equations (2.1) we get

$$\left(E_2\right)_\infty = \frac{1}{q_2}\left\{a_2 + \alpha_{21}\left(N_1\right)_\infty - \alpha_{23}\left(N_3\right)_\infty - \alpha_{22}\frac{c_2}{p_2 q_2}\right\} \tag{6.13}$$

Now $\left(E_2\right)_\infty > 0$, where the following condition

$$\left(a_2 + \alpha_{21}\left(N_1\right)_\infty\right) > \alpha_{23}\left(N_3\right)_\infty + \alpha_{22}\frac{c_2}{p_2 q_2} \tag{6.14}$$

Case (iii): if $c_1 > p_1 q_1 N_1$, $c_2 > p_2 q_2 N_3$ then the cost is greater than the revenue for the both species and whole species will be closed.

Case (iv): if $c_1 < p_1 q_1 N_1$, $c_2 < p_2 q_2 N_3$ the cost is less than revenue return on the harvesting of the both the species, the system becomes operational to yield profit.

The bionomic equilibrium $\left((N_1)_\infty, (N_2)_\infty, (N_3)_\infty, \left(E_1\right)_\infty, \left(E_2\right)_\infty\right)$ is the positive solution, solving the system (6.1 - .6.4), we get

$$\left(N_1\right)_\infty = \frac{c_1}{p_1 q_1}, \left(N_2\right)_\infty = \frac{c_2}{p_2 q_2}, \left(N_3\right)_\infty = \frac{1}{\alpha_{33}}\left(a_3 - \alpha_{31}\frac{c_1}{p_1 q_1} - \alpha_{32}\frac{c_2}{p_2 q_2}\right) \tag{6.15}$$

$$\left(E_1\right)_\infty = \frac{1}{q_1}\left\{a_1 - \alpha_{11}\frac{c_1}{p_1 q_1} - \alpha_{12}\left(N_2\right)_\infty - \alpha_{13}\left(N_3\right)_\infty\right\} \tag{6.16}$$

$$\left(E_2\right)_\infty = \frac{1}{q_2}\left\{a_2 + \alpha_{21}\left(N_1\right)_\infty - \alpha_{23}\left(N_3\right)_\infty - \alpha_{22}\frac{c_2}{p_2 q_2}\right\} \quad (6.17)$$

$$\left(E_1\right)_\infty > 0,\ \left(E_2\right)_\infty > 0$$

$$a_1 > \left(\alpha_{11}\frac{c_1}{p_1 q_1} + \alpha_{12}\frac{c_2}{p_2 q_2} + \alpha_{13}\left(N_3\right)_\infty\right),$$

$$\left(a_2 + \alpha_{21}\frac{c_1}{p_1 q_1}\right) > \alpha_{23}\left(N_3\right)_\infty + \alpha_{22}\frac{c_2}{p_2 q_2} \quad (6.18)$$

Thus the bionomic equilibrium $\left((N_1)_\infty, (N_2)_\infty, (N_3)_\infty, \left(E_1\right)_\infty, \left(E_2\right)_\infty\right)$ exits if the conditions (6.17) and (6.18) hold.

7 .OPTIMAL HARVESTING POLICY

In this section we discussed the optimal policy of harvesting of the prey and predator species. Our aim is to select the harvesting policy that maximises the present value J of continuous time stream of revenues given by

$$J = \int_0^\infty e^{-\delta t}\left[(p_1 q_1 N_1 - c_1)E_1(t) + (p_2 q_2 N_2 - c_2)E_2(t)\right]dt \quad (7.1)$$

where δ denotes the instantaneous annual rate of discount. Intentionally we have to maximize (7.1) subject to the state equations (2.1) by applying Pontryagin's maximum principle.

The control variable $E_i(t)$ is subjected to the constrains $0 \le E_i(t) \le (E)_{\max}$.

The Hamiltonian for the problem is given by

$$\begin{aligned} H = e^{-\delta t}&\left\{\left[(p_1 q_1 N_1 - c_1)E_1\right] + \left[(p_2 q_2 N_2 - c_2)E_2\right]\right\} + \lambda_1\left[a_1 N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1 N_2 - \alpha_{13}N_1 N_3 - q_1 E_1 N_1\right] \\ &+ \lambda_2\left[a_2 N_2 + \alpha_{21}N_1 N_2 - \alpha_{22}N_2^2 - \alpha_{23}N_2 N_3 - q_2 E_2 N_2\right] + \lambda_3\left[a_3 N_3 - \alpha_{31}N_3 N_1 - \alpha_{32}N_3 N_2 - \alpha_{33}N_3^2\right] \end{aligned} \quad (7.2)$$

where $\lambda_1,$ λ_2 and λ_3 are the ad joint variables. And

Let

$$\mu_1(t) = e^{-\delta t}\left[(p_1 q_1 N_1 - c_1)E_1\right] - \lambda_1 q_1 E_1 N_1 \,. \tag{7.3}$$

and

$$\mu_2(t) = e^{-\delta t}\left[(p_2 q_2 N_3 - c_2)E_2\right] - \lambda_2 q_2 E_2 N_2 \tag{7.4}$$

are called switching functions since Hamiltonian H is linear in the control variable $E_1(t)$ and $E_2(t)$.The optimal control will be a combination of extreme controls and the singular control.

The optimal control $E_1(t)$ and $E_2(t)$ that maximizes H must satisfy the following conditions.

$$E_1 = (E_1)_{\max,} \quad \text{when } \mu_1(t) > 0 \text{ i.e } \lambda_1(t)e^{\delta t} < p_1 - \frac{c_1}{q_1 N_1} \tag{7.5}$$

$$E_1 = 0 \quad \text{when } \mu_1(t) < 0 \quad \text{i.e } \lambda_1(t)e^{\delta t} > p_1 - \frac{c_1}{q_1 N_1} \tag{7.6}$$

and

$$E_2 = (E_2)_{\max,} \quad \text{when } \mu_2(t) > 0 \text{ i.e } \lambda_2(t)e^{\delta t} < p_2 - \frac{c_2}{q_2 N_3} \tag{7.7}$$

$$E_2 = 0 \quad \text{when } \mu_2(t) < 0 \quad \text{i.e } \lambda_2(t)e^{\delta t} > p_2 - \frac{c_2}{q_2 N_2} \tag{7.8}$$

Thus the optimal harvesting policy is

$$E_1(t) = \begin{cases} (E_1)_{\max} & ; \mu_1(t) > 0 \\ 0 & ; \mu_1(t) < 0 \\ E^* & ; \mu_1(t) = 0 \end{cases} \tag{7.9}$$

and

$$E_2(t) = \begin{cases} (E_2)_{\max} & ; \mu_2(t) > 0 \\ 0 & ; \mu_2(t) < 0 \\ E^* & ; \mu_2(t) = 0 \end{cases} \tag{7.10}$$

By Pontryagin's maximum principle,

$$\frac{\partial H}{\partial E_1}=0\,;\frac{\partial H}{\partial E_2}=0\;;\;\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial N_1}\;;\;\frac{d\lambda_2}{dt}=-\frac{\partial H}{\partial N_2}\quad\text{and}\quad\frac{d\lambda_3}{dt}=-\frac{\partial H}{\partial N_3} \tag{7.11}$$

$$\frac{\partial H}{\partial E_1}=0\Rightarrow e^{-\delta t}\left[p_1q_1N_1-c_1\right]+\lambda_1\left[-q_1N_1\right]=0\Rightarrow\lambda_1=e^{-\delta t}\left[p_1-\frac{c_1}{q_1N_1}\right] \tag{7.12}$$

$$\frac{\partial H}{\partial E_2}=0\Rightarrow e^{-\delta t}\left[p_2q_2N_2-c_2\right]+\lambda_2\left[-q_2N_2\right]=0\Rightarrow\lambda_2=e^{-\delta t}\left[p_2-\frac{c_2}{q_2N_2}\right]$$
(7.13)

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial N_1}=-\left[e^{-\delta t}p_1q_1E_1+\lambda_1\{-\alpha_{11}N_1\}+\lambda_2\{\alpha_{21}N_2\}+\lambda_3\{-\alpha_{31}N_3\}\right] \tag{7.14}$$

$$\frac{d\lambda_2}{dt}=-\frac{\partial H}{\partial N_2}=-\left[e^{-\delta t}p_2q_2E_2+\lambda_1\{-\alpha_{12}N_1\}+\lambda_2\{-\alpha_{22}N_2\}+\lambda_3\{-\alpha_{32}N_3\}\right] \tag{7.15}$$

$$\frac{d\lambda_3}{dt}=-\frac{\partial H}{\partial N_3}=-\left[\lambda_1\{-\alpha_{13}N_1\}+\lambda_2\{-\alpha_{23}N_2\}+\lambda_3\{-\alpha_{33}N_3\}\right] \tag{7.16}$$

After simplification we get

$$\frac{d\lambda_1}{dt}=-e^{-\delta t}p_1q_1E_1+\lambda_1\{\alpha_{11}N_1\}-\lambda_2\{\alpha_{21}N_2\}+\lambda_3\{\alpha_{31}N_3\} \tag{7.17}$$

$$\frac{d\lambda_2}{dt}=-e^{-\delta t}p_2q_2E_2+\lambda_1\{\alpha_{12}N_1\}+\lambda_2\{\alpha_{22}N_2\}+\lambda_3\{\alpha_{32}N_3\} \tag{7.18}$$

$$\frac{d\lambda_3}{dt}=\lambda_1\{\alpha_{13}N_1\}+\lambda_2\{\alpha_{23}N_2\}+\lambda_3\{\alpha_{33}N_3\} \tag{7.19}$$

From (7.12), (7.13) and (7.18)

$$\frac{d\lambda_3}{dt}-\lambda_3\alpha_{33}N_3=\alpha_{13}N_1\left[p_1-\frac{c_1}{q_1N_1}\right]e^{-\delta t}+\alpha_{23}N_2\left[p_2-\frac{c_2}{q_2N_2}\right]e^{-\delta t}$$

i.e, $\frac{d\lambda_3}{dt} - \alpha_{33}N_3\lambda_3 = \left(A_1\right)e^{-\delta t}$ (7.20)

where $A_1 = \alpha_{13}N_1\left[p_1 - \frac{c_1}{q_1N_1}\right] + \alpha_{23}N_2\left[p_2 - \frac{c_2}{q_2N_2}\right]$

The solution of which can be obtained as $\lambda_3 = \frac{-A_1}{\left(\delta + \alpha_{33}N_3\right)}e^{-\delta t}$ (7.21)

From (7.21) and (7.17),

$$\frac{d\lambda_1}{dt} = -e^{-\delta t}p_1q_1E_1 + \lambda_1\alpha_{11}N_1 - \alpha_{21}N_2\left[p_2 - \frac{c_2}{p_2N_2}\right] + \frac{-A_1\alpha_{31}N_3}{\left(\delta + \alpha_{33}N_3\right)}e^{-\delta t}$$

i.e, $\frac{d\lambda_1}{dt} - \lambda_1\alpha_{11}N_1 = -\left(A_2\right)e^{-\delta t}$ (7.22)

The solution of which can be obtained as $\lambda_1 = \frac{A_2}{\left(\delta + \alpha_{11}N_1\right)}e^{-\delta t}$ (7.23)

where $A_2 = \left(p_1q_1E_1 + \alpha_{21}N_2\left(p_2 - \frac{c_2}{p_2N_2}\right) + \frac{A_1\alpha_{31}N_3}{\left(\delta + \alpha_{33}N_3\right)}\right)$

From (7.21) and (7.19),

$$\frac{d\lambda_2}{dt} - \alpha_{22}N_2\lambda_2 = -e^{-\delta t}p_2q_2E_2 + \alpha_{12}N_1e^{-\delta t}\left(p_1 - \frac{c_1}{p_1N_1}\right) + \frac{-A_1\alpha_{32}N_3}{\left(\delta + \alpha_{33}N_3\right)}e^{-\delta t}$$

i.e, $\frac{d\lambda_2}{dt} - \alpha_{22}N_2\lambda_2 = -\left(A_3\right)e^{-\delta t}$ (7.24)

The solution of which can be obtained as $\lambda_2 = \frac{A_3}{\left(\delta + \alpha_{22}N_2\right)}e^{-\delta t}$ (7.25)

where $A_3 = \left(p_2q_2E_2 - \alpha_{12}N_1\left(p_1 - \frac{c_1}{p_1N_1}\right) + \frac{A_1\alpha_{32}N_3}{\left(\delta + \alpha_{33}N_3\right)}\right)$

From (7.12) and (7.23), we get a singular path, $\frac{A_2}{(\delta+\alpha_{11}N_1)}e^{-\delta t}=e^{-\delta t}\left[p_1-\frac{c_1}{q_1N_1}\right]$ from which we obtain

$$\frac{A_2}{(\delta+\alpha_{11}N_1)}=\left(p_1-\frac{c_1}{q_1N_1}\right) \tag{7.26}$$

From (7.13) and (7.25), we get a singular path,

$\frac{A_3}{(\delta+\alpha_{22}N_2)}e^{-\delta t}=e^{-\delta t}\left[p_2-\frac{c_2}{q_2N_2}\right]$ from which we obtain

$$\frac{A_3}{(\delta+\alpha_{22}N_2)}=\left(p_2-\frac{c_2}{q_2N_2}\right) \tag{7.27}$$

Thus from (7.13) and (7.14), we write as,

$$F(N_1)=\left(p_1-\frac{c_1}{q_1N_1}\right)-\frac{A_2}{(\delta+\alpha_{11}N_1)}=0 \tag{7.28}$$

$$G(N_2)=\left(p_2-\frac{c_2}{q_2N_2}\right)-\frac{A_3}{(\delta+\alpha_{22}N_2)}=0 \tag{7.29}$$

There exists a unique positive root $N_1=(N_1)_\delta$ of $F(N_1)=0$ in the interval $0<(N_1)_\infty<K_1$ If the following inequalities hold: $F(0)<0, F(K_1)>0$, $F^{'}(N_1)>0$ for $N_1>0$.Similarly There exists a unique positive root $N_3=(N_3)_\delta$ if $G(N_3)=0$ in the interval $0<(N_3)_\infty<K_3$ If the following inequalities hold: $G(0)<0, G(K_3)>0$, $G^{'}(N_3)>0$ for $N_3>0$

For $N_1=(N_1)_\infty, N_3=(N_3)_\infty$

we get

$$(N_3)_\infty=\frac{1}{\alpha_{33}}\left(a_3-\alpha_{31}\frac{c_1}{p_1q_1}-\alpha_{32}\frac{c_2}{p_2q_2}\right) \tag{7.30}$$

$$\left(E_1\right)_\infty = \frac{1}{q_1}\left(a_1 - \alpha_{11}\frac{c_1}{p_1 q_1} - \alpha_{12}\frac{c_2}{p_2 q_2} - \alpha_{13}\left(N_3\right)_\infty\right) \tag{7.31}$$

$$\left(E_2\right)_\infty = \frac{1}{q_2}\left(a_2 + \alpha_{21}\frac{c_1}{p_1 q_1} - \alpha_{22}\frac{c_2}{p_2 q_2} - \alpha_{23}\left(N_3\right)_\infty\right) \tag{7.32}$$

Hence once the optimal equilibrium $\left(\left(N_1\right)_\delta, \left(N_2\right)_\delta, \left(N_3\right)_\delta\right)$ is determined, the optimal harvesting effort $\left(E_1\right)_\infty$ and $\left(E_2\right)_\infty$ can be determined. From (7.21), (7.23) and (7.25) we observe that $\lambda_i(t)e^{\delta t}\,(i=1,2,3)$ is independent of time is an optimum equilibrium. Hence they satisfy the transversality condition at ∞. That is they remain bounded as $t \to \infty$.

From (7.26) and (7.27) we have

$$\frac{A_2}{\left(\delta + \alpha_{11} N_1\right)} = \left(p_1 - \frac{c_1}{q_1 N_1}\right) \to 0 \qquad \text{as } t \to \infty$$

$$\frac{A_3}{\left(\delta + \alpha_{22} N_2\right)} = \left(p_2 - \frac{c_2}{q_2 N_2}\right) \to 0 \qquad \text{as } t \to \infty$$

Thus the total economic revenue

$$\left((N_1)_\infty, (N_2)_\infty, (N_3)_\infty, \left(E_1\right)_\infty, t\right) = 0$$

$$\left((N_1)_\infty, (N_2)_\infty, (N_3)_\infty, \left(E_2\right)_\infty, t\right) = 0$$

This implies that an infinite discount rate leads to the total economic revenue tending to zero, and hence the system would remains closed.

8. NUMERICAL EXAMPLE:

1. Let a_1=1.5; a_2=2.65; a_3=3.45; α_{11}=0.1; α_{12}=0.3; α_{13}=0.01; α_{22}=0.2; α_{21}=0.3; α_{23}=0.2; α_{33}=0.2; α_{31}=0.01; α_{32}=0.2; q_1=q_2=0.01 and E_1=E_2=10.

The graphs shows the variation with initial strengths 10, 8, 25 of prey, predator and competitor populations respectively

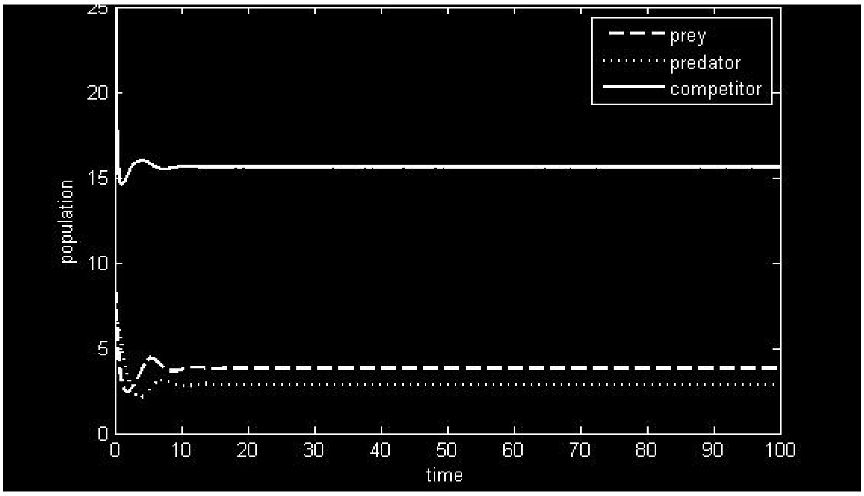

Fig 8.1.1: The variation of N_1, N_2 and N_3 with respective Time (t) for system of Eq (2.1)

From the above graph N_1, N_2 & N_3 converges with diminishing amplitude tends to equilibrium points.

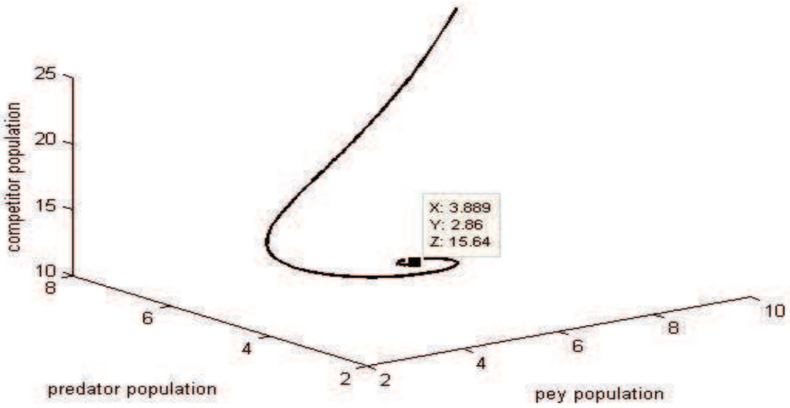

Fig 8.1.2: the phase portrait of N_1, N_2 and N_3 for system of Eq (2.1)

The above graph shows the N_1, N_2 & N_3 phase portrait. The curve is concentric spiral and the system of equations (2.1) for the given parametric values it is globally asymptotically stable and converges to equilibrium point E (3.889, 2.86, 15.64) for the system of equations (2.1) for the different initial population sizes with harvesting effort E_1=10 and E_2=10 respectively.

2. Let a_1=1; a_2=1; a_3=1.5; α_{11}=0.2; α_{12}=0.1; α_{13}=0.1; α_{22}=0.2; α_{21}=.1; α_{23}=0.1; α_{33}=0.2; α_{31}=0.1; α_{32}=0.2; q_1=q_2=0.01 and E_1=E_2=10.

The above graph shows the variation with initial strengths 20, 20, 10 of prey, predator and competitor populations respectively

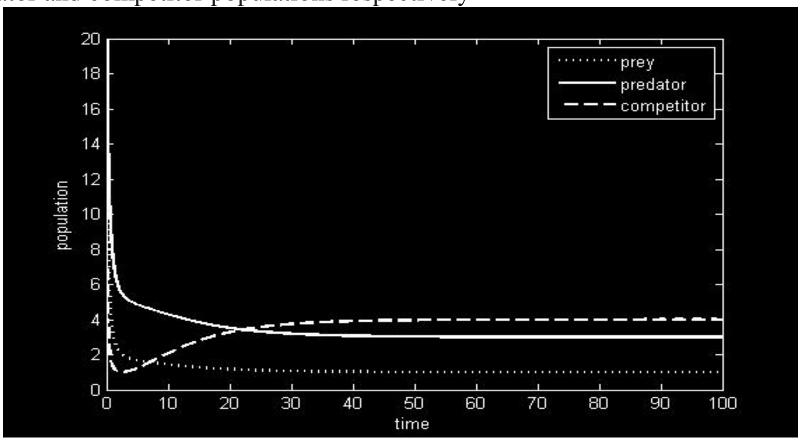

Fig 8.2.1: The variation of N_1, N_2 and N_3 with respective Time (t) for system of Eq (2.1)

From the above graph N_1, N_2 & N_3 converges. As time goes on increases the population sizes tends to equilibrium points.

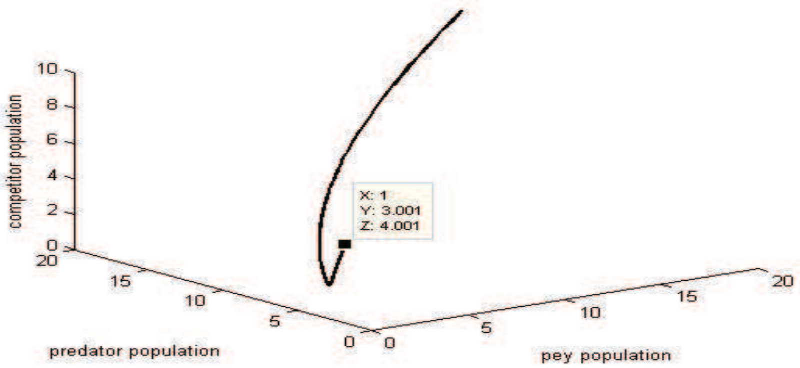

Fig 8.2.2: the phase portrait of N_1, N_2 and N_3 for system of Eq (2.1)

The above graph shows the N1, N2 & N3 phase portrait. The curve is globally asymptotically stable to equilibrium point E (1.0 3.001, 4.001) for the system of equations (2.1) for the different initial population sizes with harvesting effort E_1=10 and E_2=10 respectively.

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3. Let a_1=1.5; a_2=2.65; a_3=3.45; α_{11}=0.1; α_{12}=0.3; α_{13}=0.01; α_{22}=0.2; α_{21}=0.3; α_{23}=0.2; α_{33}=0.2; α_{31}=0.01; α_{32}=0.2; q_1=q_2=0.01

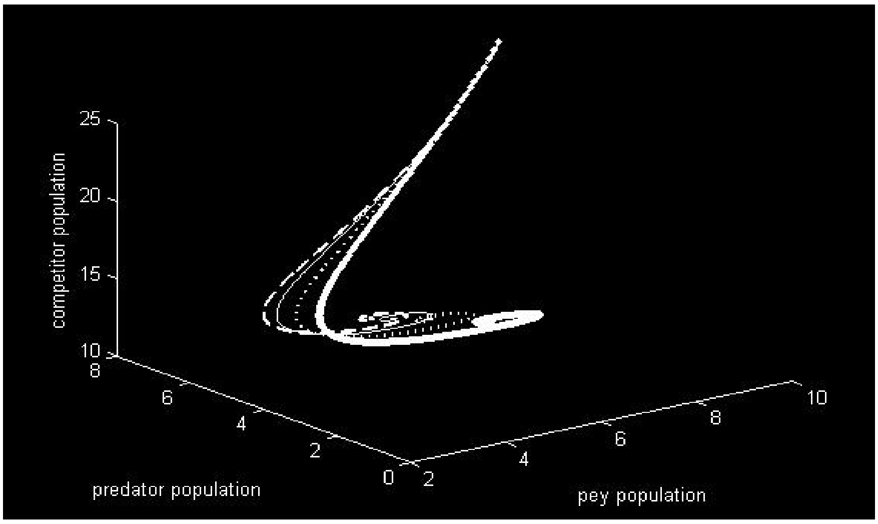

Fig 8.3.1: The phase portrait for different efforts of system of Eq (2.1)

For different efforts E1 &E2 the phase portrait are drawn

The first line with effort E1=E2=1, the second line with effort E1=E2=10

The third line with effort E1=E2=25, the fourth line with effort E1=E2=50

The above graph shows the N_1, N_2 & N_3 phase portrait. These curves are concentric spirals for the system of equations (2.1) for the given parametric values it is globally asymptotically stable and converges to equilibrium point. As the effort increase the equilibrium points are slightly vary. As $(a_1 - q_1E_1) \to 0$ and $(a_2 - q_2E_2) \to 0$ the stability of the system slowly becomes unstable.

For same effort different initial populations the phase portrait are drawn as follows with the following parametric values

3. Let a_1=1.5; a_2=2.65; a_3=3.45; α_{11}=0.1; α_{12}=0.3; α_{13}=0.01; α_{22}=0.2; α_{21}=0.3; α_{23}=0.2; α_{33}=0.2; α_{31}=0.01; α_{32}=0.2; q_1=q_2=0.01; q_1=q_2=0.01 and E_1=E_2=1.

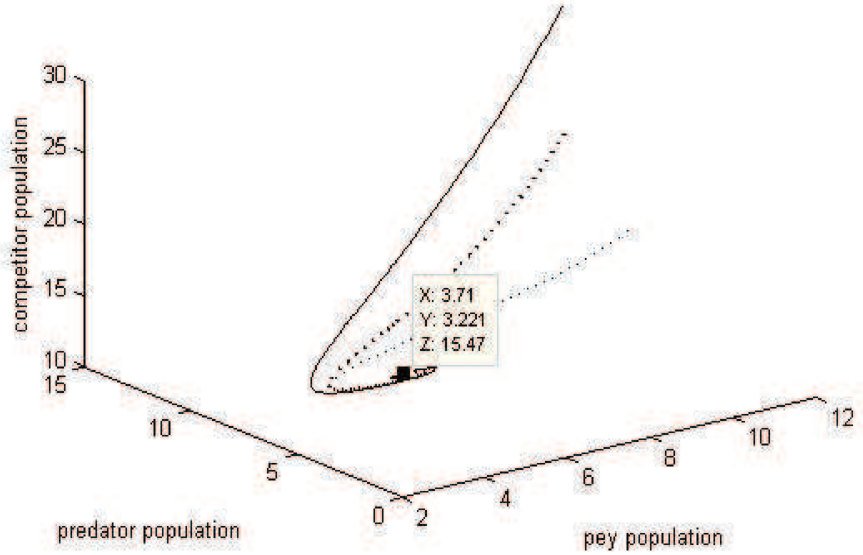

Fig 8.3.2: The phase portrait for different initial population sizes of system of Eq (2.1)

First line with initial population sizes N_1 = 10, N_2 =8, N_3 =25.Second line with initial population sizes

The above graph shows the N_1, N_2 & N_3 phase portrait for the effort E_1=E_2= 1 and for different initial population sizes as shown above. These curves are concentric spirals for the system of equations (2.1) for the given parametric values it is globally asymptotically stable and converges to equilibrium point E (3.71, 3.221, 15.47).

9. CONCLUSION

A mathematical model is proposed with a prey, predator and a competitor to the both prey and predator. The prey and predator species is subjected to continuous harvesting. From figures 8.1.1 to 8.2.2 we find that the given biological system is stable, wherein the prey and predator species are harvested. The existence of local and global stabilities is discussed. The optimal harvesting policy has been developed using Pontryagin's Maximum Principle. At the steady state level, the harvesting cost per unit effort is equal to the marginal profit of the effort. It has been found that even under continuous harvesting of the prey, the population may be maintained at an appropriate equilibrium level and infinite discount rate leads to the total economic revenue tending to zero, and hence the system would remains closed. The effect of harvesting on stability has been discussed for different values of efforts using Numerical simulation and for different initial population sizes also. From fig 8.3.1 and 8.3.2 we find that system is stable.

10. REFERENCES

1. Freedman H. I., Deterministic Mathematical Models in Population Ecology, Decker, New York, 1980.
2. Kapur J.N., *Mathematical Modeling in Biology and Medicine*. Affiliated east west, 1985.
3. Kapur J.N., *Mathematical modeling*, wiley, Easter, 1985.
4. Paul Colinvaux, *Ecology,* John Wiley and Sons Inc., New York, 1977.
5. Lotka A.J., *Elements of Physical Biology*, Williams and Wilkins, Baltimore, NewYork, 1925.
6. Volterra V, *Leçons sur la théorie mathématique de la lutte pour la vie*, Gauthier-Villars, Paris, 1931.
7. Shiva Reddy. K, Lakshmi Narayan. K, and Pattabhiramacharyulu, N.Ch. "*A Three Species Eco system consisting of A Prey, Predator and Super Predator"*. International Journal of Mathematics Applied in Science and Technology, Volume 2 , No.1 (2010), Pp.95-107.
8. Paparao. A. V and Lakshmi Narayan. K. '*A prey –predator model with a cover linearly varying with the prey population and an alternative food for the predator* Int .J. of Open Prob. Comp. & Math. (vol2, No: 2, June 2009) u_{20}
9. Chaudhuri.K.S, '*Abioeconomic Model of harvesting of multi species fishery EcoModel''* 1986,32 :267-279, doi, 10.1016/0304-3800(86)90091-8.
10. T.K.Kar, Chaudhuri.K.S''*Harvesting in two Prey one predator fishery a Bionomic model ''*.ANZIAMJ.2004,45 :443-456. u_{20}
11. Shiva Reddy. K, Pattabhiramacharyulu, N.Ch. ''*A three species eco systems with an optimal harvesting of the prey ''*: journal of Advanced Research In: vol 00 No.pp.1-11 Febrary 2012
12. Clark C.W, *mathematical bioeconomics* : The optimal management of renewable Resource (wiley, New York ,1976)
13. Ragozin.D.L and Brown.G, '*Harvesting policies and non-market valuation in a predator Prey system* ', J.Envirn .E con .Manag .12 (1985) 155-168.
14. Dubey.B, Peeyush Chandra, Prawal sinha. ''*A model of fishary resourse with reserve area'' : Non linear analysis ''*: Real world applications : 2003,4 :625-637

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