A THREE SPECIES ECOLOGICAL MODEL WITH A PREY, PREDATOR AND A COMPETITOR TO THE PREDATOR

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***Abstract:** The present paper is devoted to an analytical investigation of three species ecological model with a Prey (N_1), a predator (N_2) and a competitor (N_3) to the Predator. Species (N_2) and species (N_3) are competing for food other than prey (N_1), which is considered as an alternative for the later two on this model. In addition to that, prey species (N_1) are provided with alternative food. The model is characterized by a set of first order non-linear ordinary differential equations. All the eight equilibrium points of the model are identified; the local and global stability the interior equilibrium state is discussed. Further exact solutions of linearized system of equations have been derived. The stability analysis is supported by Numerical simulation using Mat lab.*

***Keywords:** Prey, Predator, Competitor, Equilibrium points, Local stability, Global stability, and Numerical examples*

1. INTRODUCTION

Ecology relates to the study of living beings (animals and plants) in relation to their habits and habitats. This discipline of knowledge is a branch of evolutionary biology purported to explain how or what extent the living beings are regulated in nature .Allied to the problem of population regulation is the problem of species distribution prey ,predator ,competition and soon. Research in the area of theoretical ecology was initiated in 1925 by Lotka [5] and by Volterra [6]. The general concept of modeling have been presented in the treatises of Paul Colinvaux [4], Freedman [1], Kapur [2,3] etc. The ecological symbiosis can be broadly classified recently as prey-predator, competitor, mutualism, commensalism, amensalism and so on .Recently Shiva Reddy [7] discussed the stability analysis three species eco system consting of prey, predator and super predator and Paparao [8] have discussed with a cover linearly varying with the prey population and an alternative food for the predator. Inspired from that, we discussed a more general three species model. The model is characterized by a set of first order ordinary differential equations. All the eight equilibrium points of the model are identified; the local stability is discussed using Routh-Hurwitz certiria and global stability criteria using Lyapunov's function. The stability analysis is supported by Numerical simulation using Mat lab.

2. BASIC EQUATIONS

The model equations for a three species Prey - Predator and competitor to the predator system is given by the following system of first order ordinary differential equations employing the following notation:

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1{}^2 - \alpha_{12} N_1 N_2$$
$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N^2 + \alpha_{21} N_1 N_2 - \alpha_{23} N_2 N_3 \qquad (2.1)$$
$$\frac{dN_3}{dt} = a_3 N_3 - \alpha_{33} N_3^2 - \alpha_{32} N_2 N_3$$

Where N_i , N_2 and N_3 are the populations of the prey and predator and a competitor to the predator with the natural growth rates a_1 , a_2 and a_3 respectively,

α_{11} is rate of decrease of the prey due to insufficient food and inter species competition,
α_{12} is rate of decrease of the prey due to inhibition by the predator,
α_{21} is rate of increase of the predator due to successful attacks on the prey,
α_{22} is rate of decrease of the predator due to insufficient food other than the prey and inter species competition,
α_{23} is rate of decrease of the predator due to the competition with the third species,
α_{33} is rate of decrease of the Competitor to the predator due to insufficient food and inter species competition ,
α_{32} is rate of decrease of the competitor to the predator due to the competition with the predator .

3. EQUILIBRIUM STATES

The system under investigation has eight equilibrium states. They are

I.E$_{1:}$ The extinct state $\overline{N_1} = 0, \overline{N_2} = 0$, $\overline{N_3} = 0$. (3.1)

II.E$_{2:}$ The state in which only the predator survives and the prey and competitor to the predator are extinct

$$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{\alpha_{22}}, \overline{N_3} = 0 . \qquad (3.2)$$

III. E$_3$: The state in which both the prey and the predators extinct and competitor to the predator survive

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$$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{\alpha_{33}}. \tag{3.3}$$

IV. E_4: The state in which both the predator and competitor to the predator extinct and prey survive

$$\overline{N_1} = \frac{a_1}{\alpha_{11}}, \overline{N_2} = 0, \overline{N_3} = 0 \tag{3.4}$$

V. E_5 :The state in which both the prey and the predators exist and competitor to the predator extinct

$$\overline{N_1} = \frac{(a_1\alpha_{22} - a_2\alpha_{12})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \overline{N_2} = \frac{(a_2\alpha_{11} + a_1\alpha_{21})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \overline{N_3} = 0$$

This case arise only when $a_1\alpha_{22} > a_2\alpha_{12}$ (3.5)

VI. E_6 :The state in which both prey and competitor to the predator exist and predator extinct,

$$\overline{N_1} = \frac{a_1}{\alpha_{11}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{\alpha_{33}} \tag{3.6}$$

VII. E_7: The state in which both predator and competitor to the predator exist and prey extinct,

$$\overline{N_1} = 0, \overline{N_2} = \frac{a_2\alpha_{33} - a_3\alpha_{23}}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}}, \overline{N_3} = \frac{a_3\alpha_{22} - a_2\alpha_{32}}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}} \tag{3.7}$$

The equilibrium state exist only when $a_2\alpha_{33} > a_3\alpha_{23}, a_3\alpha_{22} > a_2\alpha_{32}$ & $\alpha_{22}\alpha_{33} > \alpha_{23}\alpha_{32}$

VIII. E_8 : The state in which prey, predator and competitor to the predator exist

$$\overline{N_1} = \frac{a_1(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) - \alpha_{12}(a_2\alpha_{33} - a_3\alpha_{23})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}\alpha_{21}\alpha_{33}}, \overline{N_2} = \frac{\alpha_{11}(a_2\alpha_{33} - a_3\alpha_{23}) + a_1\alpha_{21}\alpha_{33}}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}\alpha_{21}\alpha_{33}}$$
$$\overline{N_3} = \frac{\alpha_{11}(a_3\alpha_{22} - a_2\alpha_{32}) + \alpha_{21}(a_3\alpha_{12} - a_1\alpha_{32})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}\alpha_{21}\alpha_{33}} \tag{3.8}$$

The equilibrium state exist only when

$$(a_1\alpha_{22}\alpha_{33}+\alpha_{12}a_3\alpha_{23})>(\alpha_{12}a_2\alpha_{33}+a_1\alpha_{23}\alpha_{32}),$$
$$(\alpha_{11}a_2\alpha_{33}+a_1\alpha_{21}\alpha_{33})>\alpha_{11}a_3\alpha_{23},(\alpha_{11}a_3\alpha_{22}+\alpha_{21}a_3\alpha_{12})>(\alpha_{11}a_2\alpha_{32}+\alpha_{21}a_1\alpha_{32}) \quad (3.9)$$
$$\&(\alpha_{11}\alpha_{22}\alpha_{33}+\alpha_{12}\alpha_{21}\alpha_{33})>\alpha_{11}\alpha_{23}\alpha_{32}$$

If these conditions satisfies all other equilibrium points becomes unstable.

4 .THE STABILITY OF THE EQUILIBRIUM STATE:

Let $N=(N_1,N_2,N_3)^T=\overline{N}+U$ (4.1)

Where $U=(u_1,u_2,u_3)^T$ is the perturbation over the equilibrium state. $\overline{N}=(\overline{N_1},\overline{N_2},\overline{N_3})^T$ .The basic equations (2.1) are linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt}=AU \quad (4.2)$$

Where

$$A=\begin{bmatrix} a_1-2\alpha_{11}N_1-\alpha_{12}N_2 & -\alpha_{12}N_1 & 0 \\ -\alpha_{21}N_2 & a_2-2\alpha_{22}N_2+\alpha_{21}N_1-\alpha_{23}N_3 & 0 \\ 0 & -\alpha_{32}N_3 & a_3-2\alpha_{33}N_3-\alpha_{32}N_2 \end{bmatrix}$$

The characteristic equation for the system is $det\left[A-\lambda I\right]=0$ (4.3)

The equilibrium state is stable, if three roots of the equation (4.3) are negative in case they are real or the roots have negative real parts in case they are complex.

4.1. Stability of the equilibrium state E_8:

First consider the local stability of the equilibrium

The variational matrix of the system (2.1) at interior equilibrium state E_8 is

$$A=\begin{bmatrix} -\alpha_{11}N_1 & -\alpha_{12}N_1 & 0 \\ \alpha_{21}N_2 & -\alpha_{22}N_2 & -\alpha_{23}N_3 \\ 0 & -\alpha_{32}N_2 & -\alpha_{33}N_3 \end{bmatrix} \quad (4.1.1)$$

The characteristic equation of interior equilibrium state $E_8(\overline{N}_1,\overline{N}_2,\overline{N}_3)$ is

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$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{4.2.1}$$

Where

$$b_1 = \alpha_{11}\bar{N}_1 + \alpha_{22}\bar{N}_2 + \alpha_{33}\bar{N}_3 \ ,$$
$$b_2 = \alpha_{11}\alpha_{33}\overline{N_1}\,\overline{N_3} + (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\overline{N_2}\,\overline{N_3} + (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\overline{N_1}\,\overline{N_2}$$
$$b_3 = (\alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{22}\alpha_{11}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32})\overline{N_1}\,\overline{N_2}\,\overline{N_3}$$

By Routh-Hurwitz criteria, when all Eigen values of the above characteristic equation have negative real parts if only if $b_1 > 0$, $\left(b_1b_2 - b_3\right) > 0$ and $b_3(b_1b_2 - b_3) > 0$ .clearly $b_1 > 0$ and $b_3 > 0$ if $(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) > 0$

And based on certain algebraic deductions applicable in this case, it can be verified that $b_3(b_1b_2 - b_3) > 0$.

Therefore the roots of (4.1.1) are real and negative or complex conjugates having negative real parts.

Thus the system of is locally stable for interior equilibrium point $E_8\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$

If $(\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{21}\alpha_{33}) > \alpha_{11}\alpha_{23}\alpha_{32}$.

4.2 .The solution of the linearized system of equations:

The solution of linearized system of equations (4.2) are given as

$$u_1 = A_1e^{s_1t} + B_1e^{s_2t} + C_1e^{s_3t} \tag{4.2.1}$$
$$u_2 = A_2e^{s_1t} + B_2e^{s_2t} + C_2e^{s_3t} \tag{4.2.2}$$
$$u_3 = A_3e^{s_1t} + B_3e^{s_2t} + C_3e^{s_3t} \tag{4.2.3}$$

Here s_1 , s_2 and s_3 are roots of equation (4.1.1)

$$A_1 = \left[\frac{u_{10}(s_1+\alpha_{22}\overline{N_2})+(s_1+\alpha_{33}\overline{N_3})-u_{20}\alpha_{12}\overline{N_1}(s_1+\alpha_{33}\overline{N_3})-\left[u_{30}\alpha_{12}\alpha_{23}\overline{N_1}\,\overline{N_2}+u_{10}\alpha_{23}\alpha_{32}\overline{N_2}\,\overline{N_3}\right]}{(s_1-s_2)(s_1-s_3)}\right]$$

$$B_1=\left[\frac{u_{10}(s_2+\alpha_{22}\overline{N_2})+(s_2+\alpha_{33}\overline{N_3})-u_{20}\alpha_{12}\overline{N_1}(s_2+\alpha_{33}\overline{N_3})-\left[u_{30}\alpha_{12}\alpha_{23}\overline{N_1}\overline{N_2}+u_{10}\alpha_{23}\alpha_{32}\overline{N_2}\overline{N_3}\right]}{(s_2-s_1)(s_2-s_3)}\right]$$

$$C_1=\left[\frac{u_{10}(s_3+\alpha_{22}\overline{N_2})+(s_3+\alpha_{33}\overline{N_3})-u_{20}\alpha_{12}\overline{N_1}(s_3+\alpha_{33}\overline{N_3})-\left[u_{30}\alpha_{12}\alpha_{23}\overline{N_1}\overline{N_2}+u_{10}\alpha_{23}\alpha_{32}\overline{N_2}\overline{N_3}\right]}{(s_3-s_1)(s_3-s_2)}\right]$$

$$A_2=\left[\frac{u_{20}(s_1+\alpha_{11}\overline{N_1})+(s_1+\alpha_{33}\overline{N_3})-u_{30}\alpha_{23}\overline{N_2}(s_1+\alpha_{11}\overline{N_1})+u_{10}\alpha_{21}\overline{N_2}(s_1+\alpha_{33}\overline{N_3})}{(s_1-s_2)(s_1-s_3)}\right]$$

$$B_2=\left[\frac{u_{20}(s_2+\alpha_{11}\overline{N_1})+(s_2+\alpha_{33}\overline{N_3})-u_{30}\alpha_{23}\overline{N_2}(s_2+\alpha_{11}\overline{N_1})+u_{10}\alpha_{21}\overline{N_2}(s_2+\alpha_{33}\overline{N_3})}{(s_2-s_1)(s_2-s_3)}\right]$$

$$C_2=\left[\frac{u_{20}(s_3+\alpha_{11}\overline{N_1})+(s_3+\alpha_{33}\overline{N_3})-u_{30}\alpha_{23}\overline{N_2}(s_3+\alpha_{11}\overline{N_1})+u_{10}\alpha_{21}\overline{N_2}(s_3+\alpha_{33}\overline{N_3})}{(s_3-s_1)(s_3-s_2)}\right]$$

$$A_3=\left[\frac{u_{30}(s_1+\alpha_{11}\overline{N_1})+(s_1+\alpha_{22}\overline{N_2})-u_{20}\alpha_{32}\overline{N_3}(s_1+\alpha_{11}\overline{N_1})+u_{30}\alpha_{12}\alpha_{21}\overline{N_1}\overline{N_2}-u_{10}\alpha_{21}\alpha_{32}\overline{N_2}\overline{N_3})}{(s_1-s_2)(s_1-s_3)}\right]$$

$$B_3=\left[\frac{u_{30}(s_2+\alpha_{11}\overline{N_1})+(s_2+\alpha_{22}\overline{N_2})-u_{20}\alpha_{32}\overline{N_3}(s_2+\alpha_{11}\overline{N_1})+u_{30}\alpha_{12}\alpha_{21}\overline{N_1}\overline{N_2}-u_{10}\alpha_{21}\alpha_{32}\overline{N_2}\overline{N_3})}{(s_2-s_1)(s_2-s_3)}\right]$$

$$C_3=\left[\frac{u_{30}(s_3+\alpha_{11}\overline{N_1})+(s_3+\alpha_{22}\overline{N_2})-u_{20}\alpha_{32}\overline{N_3}(s_3+\alpha_{11}\overline{N_1})+u_{30}\alpha_{12}\alpha_{21}\overline{N_1}\overline{N_2}-u_{10}\alpha_{21}\alpha_{32}\overline{N_2}\overline{N_3})}{(s_3-s_1)(s_3-s_2)}\right]$$

Where u_{10} , u_{20} and u_{30} are the initial strengths of u_1, u_2 and u_3 respectively

5. GLOBAL STABILITY:

Theorem: The Equilibrium point $E_8(\overline{N}_1, \overline{N}_2, \overline{N}_3)$ is globally asymptotically stable.

Proof: Let us consider the following Lyapunov function

$$V(\overline{N}_1,\overline{N}_2,\overline{N}_3)=\left\{N_1-\overline{N}_1-\overline{N}_1\ln\left[\frac{N_1}{\overline{N}_1}\right]\right\}+\left\{N_2-\overline{N}_2-\overline{N}_2\ln\left[\frac{N_2}{\overline{N}_2}\right]\right\}+\left\{N_3-\overline{N}_3-\overline{N}_3\ln\left[\frac{N_3}{\overline{N}_3}\right]\right\} \quad (5.1)$$

Differentiating V w.r.to ‘t’ we get

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$$\frac{dV}{dt} = \left(\frac{N_1 - \overline{N}_1}{N_1}\right)\frac{dN_1}{dt} + \left(\frac{N_2 - \overline{N}_2}{N_2}\right)\frac{dN_2}{dt} + \left(\frac{N_3 - \overline{N}_3}{N_3}\right)\frac{dN_3}{dt} \quad (5..2)$$

$$\frac{dV}{dt} < -\left(\alpha_{11} + \frac{1}{2}[\alpha_{12} - \alpha_{21}]\right)\left[N_1 - \overline{N}_1\right]^2 - \left(\alpha_{22} + \frac{1}{2}[\alpha_{12} - \alpha_{21} + \alpha_{32} + \alpha_{23}]\right)\left[N_2 - \overline{N}_2\right]^2 - \left(\alpha_{33} + \frac{1}{2}[\alpha_{32} + \alpha_{23}]\right)\left[N_3 - \overline{N}_3\right]^2$$
(5.3)

$$\frac{dV}{dt} < 0$$

Therefore , the interior equilibrium point $E_8(\overline{N}_1, \overline{N}_2, \overline{N}_3)$ is globally asymptotically stable,

If $\left(\alpha_{11} + \frac{1}{2}[\alpha_{12} - \alpha_{21}]\right)$ & $\left(\alpha_{22} + \frac{1}{2}[\alpha_{12} - \alpha_{21} + \alpha_{32} + \alpha_{23}]\right)$ are positive.

6. NUMERICAL EXAMPLE:

Let a_1=2; a_2=3; a_3=4; α_{11}=0.1; α_{12}=0.2; α_{22}=0.1; α_{21}=.3; α_{23}=0.2; α_{33}=0.2; α_{32}=.1

The graphs 6.1.1 and 6.1.2 shows the variation with initial population sizes 10, 15, 25 of prey, predator and competitor respectively.

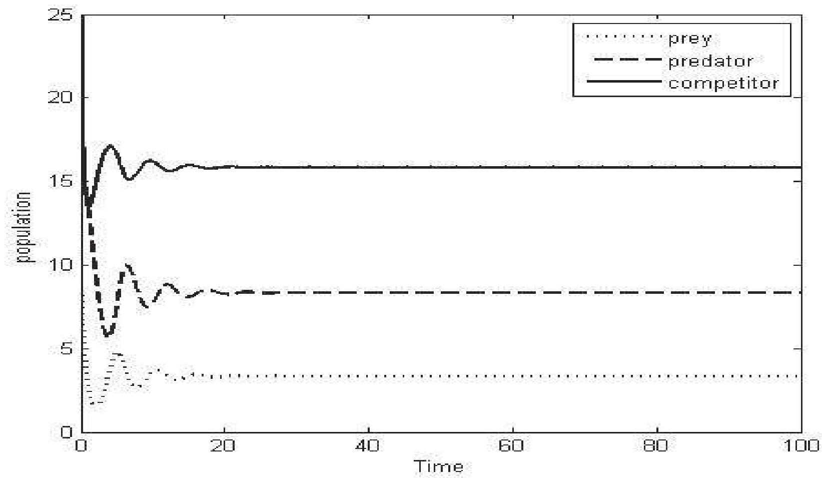

Fig 6.1.1: The Variation of N1, N2 & N3 with respective Time (t) for system of Eq (2.1)

From the above graph N1, N2 & N3 converges with diminishing amplitude as time goes on increases the population size tends to equilibrium points.

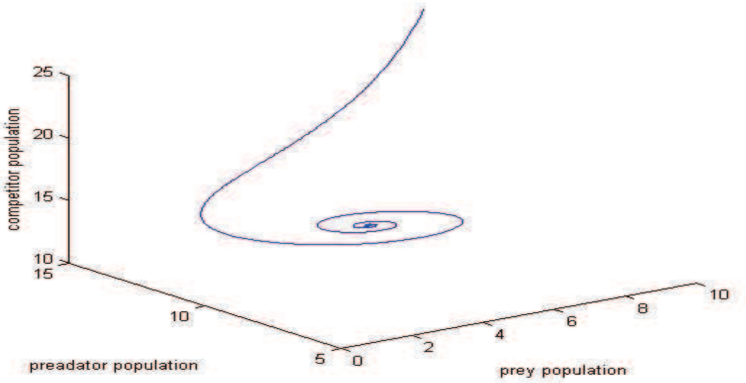

Fig 6.1.2: The Phase portrait of N1, N2, N3 for system of Eq (2.1)

The above graph shows the N1, N2 & N3 phase portrait. The curve is concentric spiral and the system of equations (2.1) for the given parametric values it is globally asymptotically stable and converges to equilibrium point E (3.33, 8.33, 15.83).

2. Let a_1=2; a_2=3; a_3=4; α_{11}=0.1; α_{12}=0.12; α_{22}=0.2; α_{21}=.13; α_{23}=0.14; α_{33}=0.3; α_{32}=.15.

The graphs 6.2.1 and 6.2.2 shows the variation with initial population sizes 40, 30, 20 of prey, predator and competitor respectively.

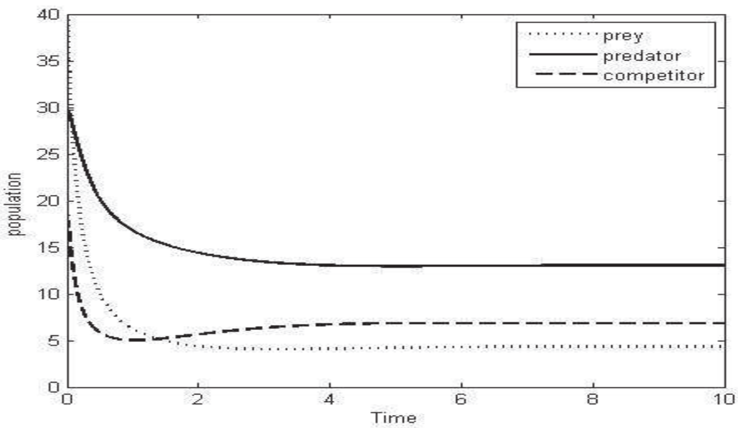

Fig 6.2.1: The Variation of N1, N2 & N3 with respective Time (t) for system of Eq (2.1)

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From the above graph N1, N2 & N3 converges. As time goes on increases the population sizes tends to equilibrium points.

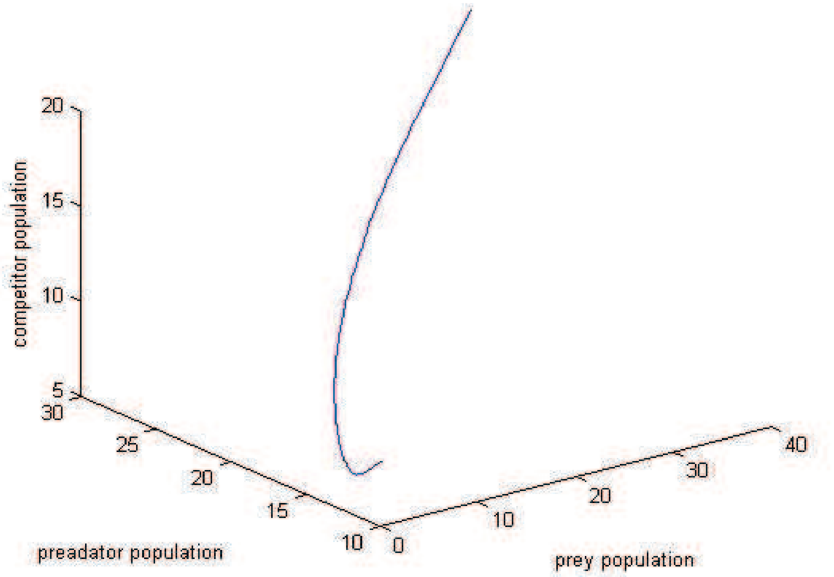

Fig 6.2.2: The Phase portrait of N1, N2, N3 for system of Eq (2.1)

The above graph shows the N1, N2 & N3 phase portrait. The curve is globally asymptotically stable to equilibrium point E (4.33, 13.05, 6.80), for the system of equations (2.1) for the given parametric values.

For the linearized system of equations exact solutions have been derived and given from equations 4.2.1, 4.2.2 and 4.2.3. The Trajectories for the solutions with the following parametric values are shown below.

3. Let $a_1=2$; $a_2=3$; $a_3=4$; $\alpha_{11}=0.1$; $\alpha_{12}=0.12$; $\alpha_{22}=0.2$; $\alpha_{21}=.13$; $\alpha_{23}=0.14$; $\alpha_{33}=0.3$; $\alpha_{32}=.15$.

The graph 6.3.1 and 6.3.2 shows the variation with initial population sizes 3, 3, 5 of prey, predator and competitor respectively.

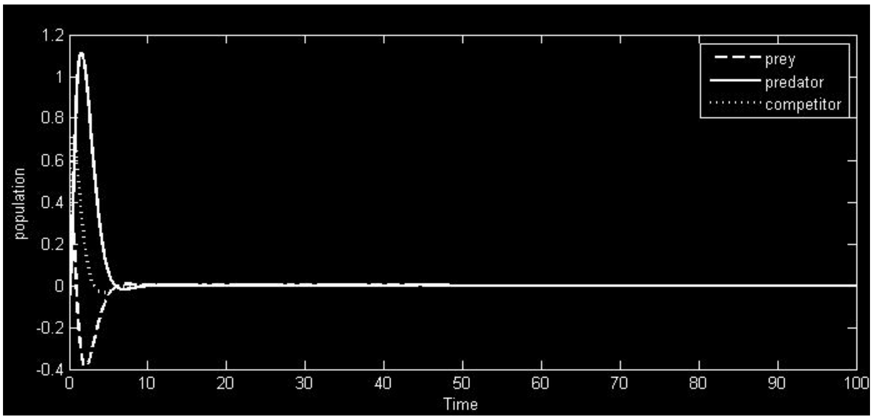

Fig 6.3.1: The Variation of u_1, u_2 & u_3 with respective Time (t) for system of Eq (4.2)

From the above graph u_1, u_2 & u_3 converges with diminishing amplitude. As time goes on increases the population sizes tends to equilibrium point.

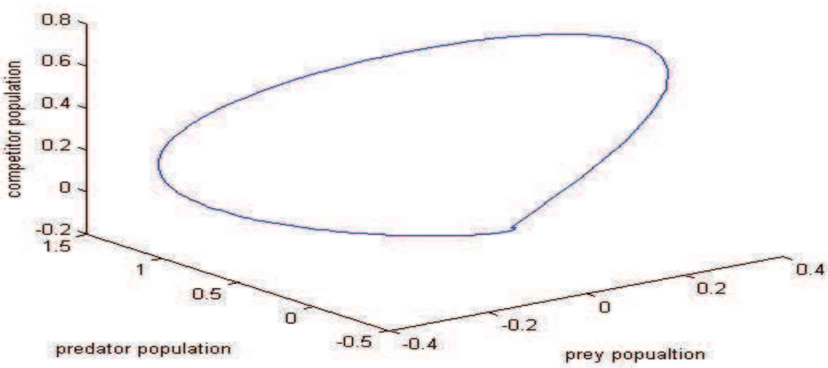

Fig 6.3.2: The phase portrait of u_1, u_2 & u_3 for system of Eq (4.2)

The above graph shows u_1, u_2 & u_3 phase portrait. The curve represents the system of equations (4.2) for the given parametric values and it converges to interior equilibrium point and hence it is globally asymptotically stable.

7. CONCLUSION

In this paper, we made an attempt to study "A three species ecological model with a prey, predator and competitor to the predator without affecting the prey species". All possible equilibrium points are identified and stability of the interior equilibrium

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point is discussed using Routh-Hurwitz criteria for local stability and using Lyapunov function for global stability. The solutions of linearized equations for interior equilibrium point are determined and are represented graphically with a suitable example. Further the non-linear systems of equations are examined for global stability with the help of suitable examples by using Mat lab. The analytical discussion and numerical examples clearly shows that the system is globally asymptotically stable.

8. REFERENCES

1. Freedman H. I., Deterministic Mathematical Models in Population Ecology, Decker, New York, 1980.
2. Kapur J.N., Mathematical Modeling in Biology and Medicine. Affiliated east west, 1985.
3. Kapur J.N., Mathematical modeling, wiley, Easter, 1985.
4. Paul Colinvaux, Ecology, John Wiley and Sons Inc., New York, 1977.
5. Lotka A.J., Elements of Physical Biology, Williams and Wilkins, Baltimore, New York, 1925.
6. Volterra V, Leçons sur la théorie mathématique de la lutte pour la vie, Gauthier-Villars, Paris, 1931.
7. Shiva Reddy. K, Lakshmi Narayan. K, and Pattabhiramacharyulu,N.Ch., “A Three Species Eco system consisting of A Prey, Predator and Super Predator”. International J.of Mathematics Applied in Science and Technology, Vol2, No.1 (2010), pp.95-107.
8. Paparao. A. V and Lakshmi Narayan. K. 'A prey –predator model with a cover linearly varying with the prey population and an alternative food for the predator’ Int .J. of Open Prob. Comp. & Math. (vol2, No: 2, June 2009)

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