ASYMPTOTIC PROPERTIES OF APPROXIMATE SOLUTIONS FOR OPTIMUM STRATIFICATION

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Abstract: Mahajan et. al. (1994) proposed the cumulative cube root rule for finding approximate optimum strata boundaries when the data on sensitive estimation variable (y) are collected by scrambled randomized response technique proposed by Eichhorn and Hayre (1983). In order to ascertain the validity of the approximate optimum strata boundaries, there is, a need to study the properties of equivalence of the approximate and exact solutions. That is what we propose to do in this paper.

Keywords: Neyman allocation, optimum stratification, regularity conditions, scrambled response

1. INTRODUCTION

Surveys on sensitive issues such as gambling, tax evasion, illegal income, sexual abuse, induced abortion, duration of suffering from AIDS, and many others, are generally affected by two serious drawbacks; respondent may not respond or deliberately give untruthful answers. This leads to substantial bias in the estimation procedures. To mitigate such bias several randomized response techniques (RRT) have been developed by researchers for collecting data on both quantitative and qualitative variables since its introduction by Warner (1965). In order to improve the precision of the estimates, developments have been focused on the use of auxiliary information at the estimation stage itself.

Mahajan *et al*. (1994) considered the problem of optimum stratification on an auxiliary variable x when the samples from different strata are selected with simple random sampling and with replacement (SSRSWR) and the data on sensitive character y are collected by scrambled randomized response technique proposed by Eichorn and Hayre (1983). It is pertinent to mention that the problem of getting reliable data on quantitative variable was first considered by Greenberg et al. (1971) and Eriksson (1973) who extended the unrelated question model (see Horvitz et al. 1967; Greenberg et al. 1969). Eichorn and Hayre (1983) proposed a scrambled randomized response method which did not contain the difficulties arise in the Greenberg et al. (1971) unrelated question method. The method involves the respondent multiplying his sensitive answer Y by a random number S from a known distribution and giving the scrambled response Z=YS to the interviewer, who does not know the particular values of the random number S.

Let the population under consideration be divided into L strata and a stratified simple random sample (SRS) of size n be drawn from it, the sample size corresponding to hth stratum being n_h so that $\sum_{h=1}^{L} n_h = n$. For hth stratum, let Y_h denotes the value of sensitive character and S_h be a scrambled random variable independent of Y_h and with finite mean and variance. The respondent generates S_h

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using some specified method and multiplies the variable value Y_h by S_h, the interviewer thus receiving the scrambled answer $Z_h = Y_h S_h$. The particular values of S_h are unknown to the interviewer, but its distribution is known. In this way, the respondents' privacy is not violated.

For theoretical development, let $E(S_h) = \theta_h$, $Y(S_h) = \gamma_h$, $E(Y_h) = \mu_{hy}$ and $V(Y_h) = \sigma^2_{hy}$ where θ_h and are γ_h are known to the interviewer but μ_{hy} and σ^2_{hy} are unknown. If Z_{hi} denote the value of the scrambled variable Z for ith unit of the sample in the hth stratum and sampling within each stratum is SRSWR, then an unbiased estimator for μ_{hy} is $\hat{\mu}_{hy} = \frac{\overline{z_h}}{\theta_h}$ where $\overline{z_h} = n_h^{-1} \sum_{i=1}^{n_h} z_{hi}$. Mahajan et al. (1994) have shown that, an unbiased estimator of population mean μ is $\hat{\mu}_{st} = \sum_{h=1}^{L} W_h \hat{\mu}_{hy}$ with a variance

$$V(\hat{\mu}_{st}) = \sum_{h=1}^{L} W_h^2 n_h^{-1} \{\sigma^2_{hy}(1 + C_h^2) + \mu^2_{hy} C_h^2\} \quad (1.1)$$

where W_h is the proportion of units falling in the hth stratum and $C_h = \frac{\sqrt{\gamma_h}}{\theta_h}$, the coefficient of varation of the scrambling variable S_h. If the cost of observing any unit in the population is assumed to be the same, the variance in (1.1) is minimized by adopting Neyman method of allocating the sample to different strata. Under this allocation, the variance in (1.1) becomes

$$V(\hat{\mu}_{st})_N = n^{-1} \left(\sum_{h=1}^{L} W_h \sqrt{\sigma^2_{hy}(1 + C_h^2) + \mu^2_{hy} C_h^2}\right)^2 \quad (1.2)$$

1.1 Minimal Equations and their Approximate Solutions

Suppose the sensitive study variable y (e.g. income understated in income tax return) and non-sensitive stratification variable x (e.g. eye estimated value of the property) be related as

$$y = c(x) + e \quad (1.3)$$

where $c(x)$ is a real valued function of x and e is the error term such that $E(e|x) = 0$ and $V(e|x) = \phi(x) > 0 \; \forall \; x \in (a, b)$ such as $(b - a) < \infty$. If f(x) is the marginal density function of x then define

$$W_h = \int_{x_{h-1}}^{x_h} f(x)dx, \;\; \mu_{hy} = \mu_{hc} = \frac{1}{W_h} \int_{x_{h-1}}^{x_h} c(x)f(x)dx \text{ and } \sigma^2_{hy} = \sigma^2_{hc} + \mu_{h\phi}$$

where (x_{h-1}, x_h) are the boundaries of the hth stratum, $\mu_{h\phi}$ and σ^2_{hc} are respectively the expected value of $\phi(x)$ and the variance of $c(x)$ in the hth stratum. Under this model, the variance in (1.2) reduces to $V(\hat{\mu}_{st})_N$

$$= n^{-1} \left(\sum_{h=1}^{L} W_h \sqrt{(\sigma^2_{hc} + \mu_{h\phi})(1 + C_h^2) + \mu^2_{hc} C_h^2}\right)^2 \quad (1.4)$$

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If $[x_h]$ denote the optimum strata boundaries on the scale of x, then minimal equations can be obtained by minimizing the variance in (1.4) with respect to x_h. Mahajan *et al.* (1994) obtained the minimal equations in this case and are given by

$$\frac{(1+C_h^2)[(c(x_h)-\mu_{hc})^2+\sigma_{hc}^2+\phi(x_h)+\mu_{h\phi}]+2C_h^2\mu_{hc}c(x_h)}{\sqrt{[(\sigma_{hc}^2+\mu_{h\phi})(1+C_h^2)+\mu_{hc}^2C_h^2]}} = \frac{(1+C_i^2)[(c(x_h)-\mu_{ic})^2+\sigma_{ic}^2+\phi(x_h)+\mu_{i\phi}]+2C_i^2\mu_{ic}c(x_h)}{\sqrt{[(\sigma_{ic}^2+\mu_{i\phi})(1+C_i^2)+\mu_{ic}^2C_i^2]}} \quad (1.5)$$

where $i = h + 1, h = 1, 2, 3, \ldots, (L - 1)$

The system of equations (1.5) involve strata parameters, which themselves are functions of the strata boundaries $[x_h]$. Due to this implicitly, it was somewhat difficult to obtain exact solutions. Therefore, they suggested Cum. $\sqrt[3]{P_1(x)}$ rule for obtaining approximate optimum strata boundaries (AOSB) on the auxiliary variable x. The rule is "If the function $P_1(x) = G_1(x).f(x)$ is bounded and possesses first two derivatives for all x in (a, b) with $(b - a) < \infty$ then for given value of L taking equal intervals on the cum. $\sqrt[3]{P_1(x)}$ yields AOSB". where

$$G_1(x) = \frac{\phi_1^2+4\phi^*c'^2}{(\phi^*)^{3/2}}, \ \phi^*=\phi + \phi C_h^2 + c^2C_h^2 \text{ and } \phi_1 = \phi^{*\prime}$$

In order to ascertain the validity of the approximate solutions (which are AOSB for the Neyman Allocation) through Cum. $\sqrt[3]{P_1(x)}$ rule, there is a need to study the properties of AOSB and to establish the equivalence of the approximate and exact solution of (1.5). This is what we propose to do in this paper.

2. PROPERTIES OF THE AOSB

The system of minimal equations (1.5) giving optimum points of stratification under Neyman Allocation in usual notations can be written as

$$K_h^2 \left(\int_{x_{h-1}}^{x_h} P_1(t)dt\,[10(K_h^2)]\right. = K_i^2 \left(\int_{x_h}^{x_{h+1}} P_1(t)dt\,[1 + 0(K_i^2)]\right. \quad (2.1)$$

where for h[th] stratum $K_h = x_h - x_{h-1}$, x_h= upper boundary, x_{h-1}= lower boundary and $O(K_h^3)$ approaches zero faster than K_h^2. Before proving the asymptotic equivalence of the approximate and exact solutions of the minimal equations, we define two function A_h and B_h of points (x_{h-1}, x_h) as

$$A_h(x_{h-1}, x_h) = \frac{(1+C_h^2)[(c(x_h)-\mu_{hc})^2+\sigma_{hc}^2+\phi(x_h)+\mu_{h\phi}]+2C_h^2\mu_{hc}c(x_h)}{\sqrt{[(\sigma_{hc}^2+\mu_{h\phi})(1+C_h^2)+\mu_{hc}^2C_h^2]}}$$

$$B_h(x_{h-1}, x_h) = \frac{(1+C_h^2)[(c(x_{h-1})-\mu_{hc})^2+\sigma_{hc}^2+\phi(x_{h-1})+\mu_{h\phi}]+2C_h^2\mu_{hc}c(x_{h-1})}{\sqrt{[(\sigma_{hc}^2+\mu_{h\phi})(1+C_h^2)+\mu_{hc}^2C_h^2]}}$$

Before we proceed further, let us impose certain regularity conditions on the functions $f(x), c(x)$ and $\square(x)$. A function $\xi(x)$ is said to belong to the class Ω if it satisfies the following conditions $0 < \xi(x)$, $\xi(x) < \infty$, $\xi(x), \xi^{'}(x)$ and $\xi^{''}(x)$ exist and are continuous in (a, b) , where $(a - b) < \infty$. We assume that the function $f(x)$ and $\square(x)$ belongs to class Ω and that the function $c(x)$ satisfies the condition (1) and (2). The symbol O is used in the sense that if two functions $I_1(x)$ and $I_2(x)$ of x are such that the ratio $\frac{I_1(x)}{I_2(x)}$ remains bounded as x tends to its limit, then $I_1(x) = O[I_2(x)]$. Also, let $[x_{hi}]$ be the solutions to the approximate system of equations (2.1) to (2.2) and denoting $E_{Li} = E_L[x_{h-1}]$, such that $E_L = \sum_{h=1}^{L} Q\,(x_{h-1}, x_h)$ where $Q\,(x_{h-1}, x_h) = K_h^2 \int_{x_{h-1}}^{x_h} P_1(t)dt\,[1 + 0(K_h^2)]$ so that limiting expression of variance $V(\hat{\mu}_{st})_N$ of the stratified estimator $\hat{\mu}_{st}$ of the population mean, with Neyman allocation of the sample to the strata can be obtained from:

$$\sqrt{n\,V(\hat{\mu}_{st})_N} = \int_a^b \sqrt{\phi^*(t)}\;f(t)dt + \frac{E_L}{96}$$

where n is the total sample size.

Let $[x_h]$ denote the exact solutions to the minimal equations (1.5). Now we state and prove the following theorem:

Theorem 2.1: If the function $P_1(x) = f(x).G_1(x) \in \Omega$ where $G_1(x) = \frac{\square_1^2 + 4\square^* C'^2}{\square^{*3/2}}$ is bounded away from zero and possess first two derivatives $\forall x \in (a, b)$ with $(b - a) < \infty$, then as $L \to \infty$.

i) $\mathrm{Sup}_{(a,b)}\left(x_{hi} - x_{(h-1)i}\right) \to 0$
ii) $\lim_{L\to\infty}\left\{\mathrm{Sup}_{(a,b)}\left|x_{hi} - x_{(h-1)i}\right|\right\} \to 0$
iii) $\lim_{L\to\infty}\left[1 - \frac{E_{Lo}}{E_{Li}}\right] \to 0$
iv) $\lim\limits_{L\to\infty} E_{Li} = \lim\limits_{L\to\infty} E_L = \frac{1}{L^2}\left[\int_a^b \sqrt[3]{P_1(t)}\;dt\right]^3$

Proof:

i) Let $K_{hi} = x_{hi} - x_{(h-1)i}$ be the h^{th} stratum width corresponding to i^{th} such system of equations. Then $\inf. K_{hi} \to 0$ as $L \to \infty$. Let us suppose that this is not true and let $\lim_{L\to\infty}(\inf. K_{hi}) = \in > 0$. Then $\sum_{h=1}^{L} K_{hi} > L_{\in}$ and there exist a value L_o of L such that for all $L > L_o, L_{\in} > (b - a)$ that is $L_{\in}$ does not lie in the interval (a, b), which is not possible. Hence $\lim_{L\to\infty} \inf.(K_{hi}) = 0$. Also suppose J(i) be the stratum corresponding to which $K_{J(i)} = \inf. K_{hi}$. If c is the constant for approximate system of equations, then $c \to 0$ as $\inf. K_{hi} = K_{j(i)} \to 0$. Since the relation, $\left(\int_{(h-1)i}^{x_{hi}} P_1(t)dt\right) = c$ holds for all h and also $0 < P_1(x) < \infty$ for all x in (a, b), it clearly shows that $\sup_{(a,b)}(K_{hi}) \to 0$ as $L \to \infty$. This proves (i)

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ii) The system of equation (2.2) can also be expressed as

$$A\,(x_{h-1}, x_h) = B(x_h, x_{h+1}) \qquad (2.3)$$

where $A(x_h, x_{h+1})$ and $B(x_h, x_{h+1})$ differ in respect of the terms $O(K^2_{h+1})$ in $[1+O(K^2_{h+1})]$. Expanding the two sides of (2.3) by the Taylor's theorem corresponding to the points $\left(x_{(h-1)i}, x_{hi}\right)$ and $(x_{hi}, x_{(h+1)i})$ respectively; the system of equations (2.3) reduces to the following form;

$$K^2_{hi}\int_{x_{(h-1)i}}^{x_{hi}} P_1(t)dt\,[1+0(K^2_{hi})] + Z_{(h-1)i}\frac{\partial A\left(V_{(h-1)i}, V_{hi}\right)}{\partial V_{(h-1)i}} + Z_{hi}\frac{\partial A\left(V_{(h-1)i}, V_{hi}\right)}{\partial V_{hi}}$$

$$= K^2_{(h+1)i}\int_{x_{hi}}^{x_{(h+1)i}} P_1(t)dt\,[1+0(K_{(h+1)i})^2] + Z_{hi}\frac{\partial B\left(V_{hi}, V_{(h+1)i}\right)}{\partial V_{hi}}$$

$$+ Z_{(h+1)i}\frac{\partial B\left(V_{hi}, V_{(h+1)i}\right)}{\partial V_{(h+1)i}} \qquad (2.4)$$

where $Z_{hi} = x_h - x_{hi}$ and $V_{ji} = x_{ji} + \delta z_{ji}, 0 \le \delta < 1$

Since $K^2_{hi}(\int_{(h-1)i}^{x_{hi}} P_1(t)dt) = K^2_{(h+1)i}(\int_{x_{hi}}^{x_{(h+1)i}} P_1(t)dt) = c$

where c is the constant of approximation for the system of equations. Equation (2.4) can be written as

$$Z_{(h-1)i}\frac{\partial A\left(V_{(h-1)i}, V_{hi}\right)}{\partial V_{(h-1)i}} + Z_{hi}\left[\frac{\partial A\left(V_{(h-1)i}, V_{hi}\right)}{\partial V_{hi}} - \frac{\partial B\left(V_{hi}, V_{(h+1)i}\right)}{\partial V_{(h+1)i}}\right] - Z_{(h+1)i}\frac{\partial B\left(V_{hi}, V_{(h+1)i}\right)}{\partial V_{(h+1)i}}$$

$$= K^2_{(h+1)i}\int_{x_{hi}}^{x_{(h+1)i}} P_1(t)dt\,[1+0(K^2_{(h+1)i})] - K^2_{hi}\int_{x_{(h-1)i}}^{x_{hi}} P_1(t)dt\,[1+0(K^2_{hi})]$$

which implies,

$$G_{(h-1)i} + 2G_{hi} - G_{(h+1)i} = Y_{hi}$$

where

$$G_{(h-1)i} = -Z_{(h-1)i}\frac{\partial A\left(V_{(h-1)i}, V_{hi}\right)}{\partial V_{(h-1)i}}$$

$$G_{hi} = Z_{hi}\left[\frac{\partial A\left(V_{(h-1)i}, V_{hi}\right)}{\partial V_{hi}} - \frac{\partial B\left(V_{hi}, V_{(h+1)i}\right)}{\partial V_{(h+1)i}}\right] \text{ and}$$

$$G_{(h+1)i} = Z_{(h+1)i}\frac{\partial B\left(V_{hi}, V_{(h+1)i}\right)}{\partial V_{(h+1)i}} \qquad (2.5)$$

$$Y_{hi} = K^2_{(h+1)i}\int_{x_{hi}}^{x_{(h+1)i}} P_1(t)dt\,[1+0(K^2_{(h+1)i})] - K^2_{hi}\int_{x_{(h-1)i}}^{x_{hi}} P_1(t)dt\,[1+0(K^2_{hi})] \qquad (2.6)$$

Now, $C^{1/3} = 0(K_{hi})$, for any h can be easily satisfied by solution.

$$G_{ji} = L^{-1}\left[(L-\propto)\sum_{h=1}^{\propto} KY_{ki} + \propto \sum_{h=\propto+1}^{L-1}(L-h)\,Y_{hi}\right],$$
$$\propto = h-1, h, h+1$$

Also,

$$K_{hi} = \left[C/M(X_{(h-1)i} + \delta_h K_{hi})\right]^{1/3}$$
$$= C^{1/3}M^{-1/3}\left[(X_{(h-1)i} + \delta_h K_{hi})\right]^{1/3}$$

Similarly, $K_{(h+1)i} = C^{1/3}M^{-1/3}(X_{hi} + \delta_{(h+1)}K_{(h+1)i})$

Therefore,

$$K_{(h+1)i} - K_{hi} = C^{1/3}\left[M^{-1/3}\left(X_{hi} + \delta_{(h+1)}K_{(h+1)i}\right) - M^{-1/3}(X_{(h-1)i} + \delta_h K_{hi})\right]$$
$$= C^{1/3}\left[M^{-1/3}\left(X_{hi} + \delta_{(h+1)}K_{(h+1)i}\right) - M^{-\frac{1}{3}}(X_{hi} - K_{hi} + \delta_h K_{hi}\right]$$
$$= C^{1/3}\left[M^{-1/3}\left(X_{hi} + \delta_{(h+1)}K_{(h+1)i}\right) - M^{-\frac{1}{3}}(X_{hi} - (1-\delta h)K_{hi}\right]$$
$$= 0(K_{hi}^2) + 0(K_{(h+1)i}^2)$$

Hence, $Y_{hi} = C^{1/3}\,0(K_{hi}^5) = 0(m_i^6)$ (2.7)

Taking $m_i = \text{Sup}_{(a,b)}(K_{hi})$ and $K = \text{constant} < \infty$, than

$|G_{hi}| = \left|C^{1/3}L^{-1}\,0\left[\sum_{h=1}^{L} K_{hi}^5\right]\right| \leq K\,0(m_i)\,0(m_i^{-1})0(m_i^{-1})\,0(m_i^5) = 0\,(m_i^4)$

Since, $\inf. K_{hi} \leq \frac{(b-a)}{L} \leq \text{Sup}_{(a,b)}K_{hi} = m_i$, so then

$L = 0\,[m_i^{-1}\,(b-a)]$ As $0(m_i^{-2})$ is the order of coefficient of $Z_{\propto i}(\propto = h-1, h, h+1)$ in (2.5), then (2.6) from, $Y_{\propto i} = Z_{\propto i}$ [Coefficient of $0\,(m_i^4)$] $= 0\,(m_i^6)$

This implies, $Z_{\propto i} = 0\,(m_i^2)$ (2.8)

Since the members of minimal equation of order $0(K_{hi}^2)$, the set $[X_{hi}]$ must therefore be adjusted by terms of order $0\left[\text{Sup}_{(a,b)}(x_{hi} - x_{(h-1)i})\right]^2$ in order to satisfy minimal equations. This proves (ii)

(iii)Let$\Delta_i = E_{Li} - E_{Lo} = \sum_{h=1}^{L} R\,(x_{(h-1)i}, x_{hi}) - \sum_{h=1}^{L} R\,(x_{h-1}, x_{hi})$ so that $\Delta_i \geq 0$, where $R(x_{h-1}, x_{hi}) = K_h^2 \int_{x_{h-1}}^{x_{hi}} P_1(t)dt\,[1 + 0(K_h^2)]$

In order to explain this part, expanding $R(x_{(h-1)i}, x_{hi})$ by Taylor theorem, we have

$$\sum_{h=1}^{L} R\,(x_{(h-1)i}, x_{hi}) = \sum_{h=1}^{L} R\,(x_{h-1}, x_{hi}) + \frac{1}{2}\sum_{h=1}^{L-1} Z_{hi}^2\left(\frac{\partial^2 R(U_{(h-1)i}, U_{hi})}{\partial^2 U_{hi}} + \frac{\partial^2 R(U_{hi}, U_{(h+1)i})}{\partial^2 U_{hi}}\right) + \sum_{h=1}^{L-1} Z_{hi}Z_{(h-1)}\frac{\partial^2 R(U_{(h-1)i}, U_{hi})}{\partial^2 U_{(h-1)i}\,\partial^2 U_{hi}}$$ (2.9)

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where, $Z_{hi} = x_{hi} - x_h$, $U_{ji} = U_j - \delta Z_{ji}$, $0 < \delta < 1$

As proved in (2.8), $Z_{hi} = 0(m_i^2)$ and $Sup_{(a,b)}(K_{hi}) = m_i$, then partial derivatives in (2.9) can easily be shown of order $0(m_i)$.

Since, inf. $(K_{hi}) \le \frac{(b-a)}{L} \le Sup_{(a,b)}(K_{hi})$, then $L = 0(m_i^{-1})$
Also, from (2.9), after simplification $\Delta_i = 0(m_i^4)$

Again $R(x_{(h-1)i}, x_{hi}) = K_h^2 \int_{x_{h-1}}^{x_{hi}} P_1(t)dt\,[1 + 0(K_{hi}^2)]$ and $0 < P_1(t) < \infty$, then $R(x_{(h-1)i}, x_{hi}) = 0(K_{hi}^2) = 0(m_i^2)$. Thus $\left[1 - \frac{E_{Li}}{E_{Lo}}\right] = \frac{\Delta_i}{E_{Li}} = \frac{0(m_i^4)}{0(m_i^2)} = 0(m_i^2)$. Consequently,

$$\left[1 - \frac{E_{Lo}}{E_{Li}}\right] = 0\left\{Sup_{(a,b)}\left(x_{hi} - x_{(h-1)i}\right)\right\}^2$$

which establishes (iii)

iv) Using cum. $\sqrt[3]{p(x)}$ rule,
$K_{hi}^2 \int_{x_{(h-1)i}}^{x_{hi}} P_1(t)dt\,[1 + 0(K_{hi}^2)] = \left[\int_{x_{(h-1)i}}^{x_{hi}} \sqrt[3]{P_1(t)}\,dt\right]^3 [1 + 0(K_{hi}^2)]$,
which implies that

$\left[\int_{x_{h-1}}^{x_{hi}} \sqrt[3]{P_1(t)}\,dt\right]^3 = C$ where C is constant. Then $\int_{x_{(h-1)}}^{x_{hi}} \sqrt[3]{P_1(t)}\,dt = C^{1/3}$ and as such

$$K_{hi}^2 \int_{x_{(h-1)i}}^{x_{hi}} \sqrt[3]{P_1(t)}\,dt\,[1 + 0(K_{hi}^2)] = C^{1/3}[1 + 0(K_{hi}^2)] \tag{2.10}$$

$$\Rightarrow \sum_{h=1}^{L} \int_{x_{(h-1)i}}^{x_{hi}} \sqrt[3]{P_1(t)}\,dt\,[1 + 0(K_{hi}^2)] = \int_a^b \sqrt[3]{P_1(t)}\,dt]$$

which implies that

$$C = \left[\frac{1}{L} \int_a^b \sqrt[3]{P_1(t)}\,dt\right]^3 [1 + 0(m_i^2)] \tag{2.11}$$

But $E_{Li} = \sum_{h=1}^{L} R\left(x_{(h-1)i}, x_{hi}\right) = \sum_{h=1}^{L} K_{hi}^2 \int_{x_{h-1}}^{x_{hi}} P_1(t)dt\,[1 + 0(K_{hi}^2)]$

$$= L \int_a^b P_1(t)dt\,[1 + 0(K_{hi}^2)] = L\,C\,[1 + 0(m_i^2)] \tag{2.12}$$

Equivalently, $E_{Li} = L\left[\frac{1}{L} \int_a^b \sqrt[3]{P_1(t)}\,dt\right]^3 [1 + 0(m_i^2)]$

$$= \frac{1}{L^2}\left[\int_a^b \sqrt[3]{P_1(t)}\,dt\right]^3 [1 + 0(m_i^2)]$$

On taking limits both sides, we have

$$\lim_{L\to\infty} E_{Li} = \frac{1}{L^2}\left[\int_a^b \sqrt[3]{P_1(t)}\,dt\right]^3$$

which proves (iv)

Thus, the asymptotic equivalence between the approximate solutions $[X_{hi}]$ and the exact solution $[X_h]$ has been established.

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