A STATISTICAL METHOD TO STUDY THE EXPONENTIAL FUNCTION

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Abstract: In this paper, a new method based on regression analysis has been suggested to estimate the value of eX where X is a positive number. The method takes advantage of the high linear correlation between X and eX in a certain small interval of length one unit. Initially, eX is estimated by a linear function (e*X) of X. Next, the performance of e*X is improved by estimating the percent error in e*X, and incorporating the same in e*X.

Keywords: Coefficient of correlation, Exponential function, Method of least squares, Regression analysis.

INTRODUCTION:

In this paper, a new method based on regression analysis has been suggested to estimate the exponential function e^X where X is a positive number. In the following, the notations e^{*X} and e^{**X} respectively denote linear regression estimator of e^X and improved linear regression estimator of e^X.

The suggested method takes advantage of the high linear correlation between X and e^X. in an interval of one unit. Initially, e^X is estimated by a linear function (e^{*X}) of X, and then a new function e^{**X} is derived by estimating percent errors in e^{*X} and incorporating the same in e^{*X}.

Theorem 1 [1]

Let X ~ U(n, n+1). Then $\rho(X, e^X) = \frac{(3-e)2\sqrt{3}}{\sqrt{(8e - 6 - 2e^2)}}$.
where ρ(Y,W) represents Karl Pearson's coefficient of correlation between Y and W.

Proof:

It may be observed that $EX = n + 0.5$, $V(X) = \frac{1}{12}$. … (1)

Now, $Ee^X = \int_n^{n+1} e^X \, dX = e^{n+1} - e^n$ … (2)

$Ee^{2X} = \int_n^{n+1} e^{2X} \, dX = \frac{1}{2}\,(e^{2n+2} - e^{2n})$ … (3)

Hence, $V(e^X) = Ee^{2X} - (Ee^X)^2$

$= \frac{1}{2}\,(e^{2n+2} - e^{2n}) - (e^{n+1} - e^n)^2$ … (4)

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[by (2) and (3)]

Also, $EX.e^X = \int_n^{n+1} X.e^X \, dX = (n+1)\, e^{n+1} - ne^n - e^{n+1} + e^n$ … (5)

Finally,

$$\rho(X, e^X) = \frac{EX.e^X - EX.Ee^X}{\sqrt{V(X).V(e^X)}}$$

$$= \frac{(3-e)2\sqrt{3}}{\sqrt{(8e - 6 - 2e^2)}} \text{ ... [by (1), (2), (4) and (5)]}$$

Proof is complete.

Note 1

It may be noted that $\rho(X, e^X)$ is independent of n, the parameter in U(n, n+1).

Note 2

On simplification, it may be seen that $\rho(X, e^X) = 0.991826839$, approximately.

Theorem 2

Let X ~ U(n , n+1).
The regression equation of e^X on X is given by
$e^X = e^n [\, 6(3 - e)\, \{X - n - 0.5\} + e - 1]$.
If we let $X = n + i$ where $i \in (0, 1)$, above regression equation reduces to
$e^X = e^n [\, 6(3 - e)\, \{i - 0.5\} + e - 1]$.

Proof:

The regression equation of e^X on X is given by

$e^X - Ee^X = \beta_{e^x,x} (X - EX)$ … (6)

where $\beta_{e^x,x}$ = regression coefficient of e^X on X = $\rho(X, e^X) \sqrt{\frac{V(e^X)}{V(X)}}$.

By using the expressions for $\rho(X, e^X)$, $V(e^X)$ and $V(X)$ from Theorem 1, we get

$\beta_{e^x,x} = 6\,(3 - e)\, e^n$.

Now, (6) reduces to $e^X - e^{n+1} + e^n = 6\,(3 - e)\, e^n (X - n - 0.5)$

$\Rightarrow e^X = e^n [\, 6(3 - e)\, \{X - n - 0.5\} + e - 1]$ … (7)

If we put $X = n + i$ where $0 < i < 1$, (7) reduces to $e^X = e^n [\, 6(3 - e)\, \{i - 0.5\} + e - 1]$.

Proof is complete.

Note 3

It is necessary to know the value of e^n to estimate e^{n+i} (using (7)), $0 < i < 1$.

Note 4

The percent error in the estimate of e^X is given by

$$\left| \frac{e^n \,[\, 6(3-e)\,\{i-0.5\} + e - 1] - e^{n+i}}{e^{n+i}} \right| \text{ X } 100$$

$$= \left| \frac{[\, 6(3-e)\,\{i-0.5\} + e - 1] - e^{i}}{e^{i}} \right| \text{ X } 100.$$

It may be noted that the percent error does not depend on n.
So, for fixed i, whether we are estimating $e^{10\ +\ i}$ or $e^{\ 1000000\ +\ i}$, the percent error remains the same.

Note 5

The estimated values (as estimated by (7)) are denoted by e^{*X} . Table I is self explanatory.

Table I

X = n + i	e^X	e^{*X}	***% error in*** e^{*X}
8.01	*3010.917112*	*2652.289551*	*11.8835*
8.05	*3133.794974*	*2853.774414*	*8.9072*
8.10	*3294.486056*	*3105.630615*	*5.7026*
8.15	*3463.379086*	*3357.486816*	*3.0273*
8.20	*3640.950427*	*3609.343018*	*0.8373*
8.25	*3827.626050*	*3861.199463*	*0.9085*
8.30	*4023.872442*	*4113.055176*	*2.2481*
8.35	*4230.180892*	*4364.911133*	*3.2171*
8.40	*4447.066377*	*4616.766602*	*3.8483*
8.45	*4675.072122*	*4868.623047*	*4.1725*
8.50	*4914.767961*	*5120.478516*	*4.2180*
8.55	*5166.753257*	*5372.334473*	*4.0113*
8.60	*5431.658102*	*5624.190430*	*3.5769*
8.65	*5710.144896*	*5876.046387*	*2.9374*
8.70	*6002.909999*	*6127.902344*	*2.1140*
8.75	*6310.685475*	*6379.758301*	*1.1260*
8.80	*6634.240922*	*6631.613770*	*0.0085*
8.85	*6974.385395*	*6883.470215*	*1.2729*
8.90	*7331.969431*	*7135.325684*	*2.6517*
8.95	*7707.887175*	*7387.181641*	*4.1309*
8.99	*8022.451712*	*7588.666504*	*5.3777*

Note 6

Table 1 suggests that the linear fit provided by (7) works reasonably well for some values of i (not for all values of i) . An improved estimator (e^{**X}) is suggested below.

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In the following, the percent errors in e^{*X} (last column of Table I) are estimated by fitting second degree polynomials separately in the intervals (0, 0.22], (0.22, 0.48], (0.48, 0.80] and (0.80, 1) of i , and incorporating the same in the linear estimator e^{*X} .

Theorem 3

If X ~ U(n, n+1), e^x may be estimated by the function e^{**X}, given by

$$e^{**X} = \begin{cases} e^{*X} + (12.64612076 - 79.70674017 .i + 103.318039 .i^2) & \text{if } 0 < i \le 0.22 \\ e^{*X} - (-10.71033549 + 63.25398062. i - 67.04879324. i^2) & \text{if } 0.22 < i \le 0.48 \\ e^{*X} - (-3.528758024 + 34.17463626. i - 37.26698422. i^2) & \text{if } 0.48 < i \le .80 \\ e^{*X} + (-6.757617729 - 7.618567704 .i + 20.08156612. i^2) & \text{if } 0.80 < i < 1 \end{cases}$$

Proof

Using the data on percent errors in e^{*X} for i = 0.01 to 0.99 in steps of 0.01, the percent errors are estimated by fitting second degree polynomials.

Case 1: $0 < i \le 0.22$

Percent error is estimated by $12.64612076 - 79.70674017. i + 103.318039. i^2$

Case 2: $0.22 < i \le 0.48$

Percentage error is estimated by $-10.71033549 + 63.25398062 .i - 67.04879324. i^2$

Case 3: $0.48 < i \le 0.80$:

Percent error is estimated by $- 3.528758024 + 34.17463626. i - 37.26698422. i^2$

Case 4: $0.80 < i < 1$:

Percent error is estimated by $- 6.757617729 - 7.618567704. i + 20.08156612. i^2$

These percent errors are added to (or subtracted from) e^{*X} according as $e^{*X} < e^X$

(or $e^{*X} > e^X$).

This leads to the estimator e^{**X}, defined in the statement of the theorem.

Proof is complete.

Note 7

In Table II, the values of percent errors in e^{*X} and e^{**X} are tabulated for comparing the performances of e^{*X} and e^{**X} .

Table II

X	***Percent Error in*** e^{*X}	***Percent Error in*** e^{**X}
8.01	*11.8835*	*1.46*
8.05	*8.9072*	*0.81*
8.10	*5.7026*	*0.35*
8.15	*3.0273*	*0.14*
8.20	*0.8373*	*0.04*
8.25	*0.9085*	*0.04*
8.30	*2.2481*	*0.06*
8.35	*3.2171*	*0.13*
8.40	*3.8483*	*0.20*
8.45	*4.1725*	*0.21*
8.50	*4.2180*	*0.23*
8.55	*4.0113*	*0.17*
8.60	*3.5769*	*0.14*
8.65	*2.9374*	*0.12*
8.70	*2.1140*	*0.09*
8.75	*1.1260*	*0.06*
8.80	*0.0085*	*0.0003*
8.85	*1.2729*	*0.05*
8.90	*2.6517*	*0.10*
8.95	*4.1309*	*0.20*
8.99	*5.3777*	*0.32*

Note 8

Table II indicates that performance of e^{**X} is better than that of e^{*X} from the point of view of estimating e^X.

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