Embeddable Near-Rings

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Abstract: In this paper we study subdirectly irreducible Boolean near-rings. First we present subdirect product of family of near-rings, subdirectly irreducible near-rings, theorem relating to that each near-ring N is isomorphic to subdirect product of subdirectly irreducible near-rings, distributive elements in near-ring, anti-distributive elements in near-ring, distributively generated near-ring. Then we study Boolean near-ring with left identity, a Boolean near-ring possess IFP, results relating to distributively generated Boolean near-rings, subdirectly irreducible distributively generated Boolean near-rings and interesting theorems like (i). Every distributively generated Boolean near-ring N is a Boolean ring, (ii).Every distributive idempotent is central, (iii). A Boolean near-ring which has multiplicative identity has all elements central, (iv). A non-trivial Boolean near-ring N contains a family of completely prime ideals with trivial intersection, (v) Boolean near-ring with identity whose every non-trivial homomorphic image contains a non-zero idempotent is additively commutative, (vi). A distributively generated Boolean near-ring N is a commutative ring,(vii). Every Boolean near-ring with identity is a Boolean ring.

Keywords : subdirectly irreducible near-rings, distributively generated Boolean near-ring,

1. INTRODUCTION

1.1 **Definition:** A non-empty set N is said to be a near-ring if in N there are defined two operations, denoted by + and · respectively such that for all n, n_1, n_2, n_3 in N:

(1). $n_1 + n_2$ is in N.

(2). $(n_1 + n_2) + n_3 = n_1 + (n_2 + n_3)$.

(3). There is an element 0 in N such that $n + 0 = 0 + n = n$.

(4). There exists an element – n in N such that $n + (-n) = 0 = (-n) + n$.

(5). $n_1 \cdot n_2$ is in N.

(6). $(n_1 \cdot n_2) \cdot n_3 = n_1 \cdot (n_2 \cdot n_3)$.

(7). $(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$ (right distributive law).

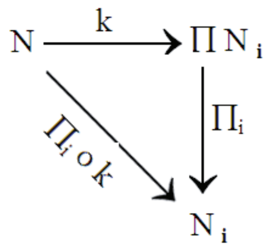
1.2 **Definition:** Suppose (N, + , ·) is a near-ring. If (N, +) is abelian we call N an abelian near-ring. If (N, ·) is commutative we call N a commutative near-ring. If $N = N_d$, N is said to be distributive. N is integral if N has no non-zero, zero-divisors. A near-ring which is not a ring will be called as a non-ring. If N*=N - {0} is a group, N is called a near-field. A near-ring with the property that N_d generates (N, +) is called a distributively generated near-ring (dgnr).

1.3 **Definition :** Suppose { N_i / i ∈ I } is a family of near- rings. The cartesian product $\underset{i \in I}{X} N_i$ with the component wise operations **+** and · is a near-ring.

It is denoted by $\pi_{i \in I} N_i$ and is called direct product of the near- rings $\{ N_i / i \in I \}$.

1.4 Definition: The sub near-ring of N_i consisting of those elements with all components equal to zero except finite number of components is called the direct sum of $\{ N_i / i \in_{i \in I}^{\pi} I \}$ and is denoted by $\oplus_{i \in I} N_i$.

1.5 Definition: Suppose $\{ N_i / i \in I \}$ is a family of near-rings. We call a near-ring N a subdirect product of $\{ N_i / i \in I \}$ if there exists a monomorphism $k : N \to \prod N_i$ such that $k_i = \prod_i o\, k$ is surjective on N_i where $\prod_i : \prod N_i \to N_i$ is an epimorphism. i.e Commutes and $\prod_i o\, k$ is surjective.



1.6 Definition : A non-zero near-ring N is subdirectly irreducible iff the intersection of all non-zero ideals of N is non-zero.

1.7 Theorem: If $\{ I_k / k \in K \}$ is a family of ideals of a near-ring N, then $\bigcup_{k \in K} I_k$ is an ideal moreover if $\{ I_k / k \in K \}$ is directed, then $\bigcup_{k \in K} I_k$ is an ideal.

1.8 Theorem: Let a, b be distinct elements of a near –ring $\bigcap_{k \in K}$N and let D(a, b) be the set of all ideals I of N such that $a - b \notin I$.
Then $D(a, b) \neq \phi$ and contains a maximal element.

1.9 Theorem: Let I be an ideal of a near-ring N and a, b be distinct elements such that $a - b \notin I$. Then the set of all ideals $J \supseteq I$ and $a - b \notin J$ contains a maximal element.

1.10 Theorem : Any proper ideal in a near-ring $N \neq 0$ is contained in a maximal ideal.

Main Theorems

2.1 Theorem: Suppose $\{ K_i / i \in I \}$ is a family of ideals of a near-ring N such that $\bigcap_{k \in K} K_i = \{ 0 \}$ then N is a subdirect product of family of sub near-rings $\{ N_i / i \in I \}$ where $N_i = N / K_i$.

2.2 **Theorem:** Each near-ring N is isomorphic to subdirect product of subdirectly irreducible near-rings.

Proof. Assume that N contains more than one element. For every a , b Э a ≠ b there is a maximal ideal M (a, b) Э a – b ∉ M (a, b).

Then N / M(a, b) is subdirectly irreducible.

And ∩ M (a, b) ≠ { 0 }.

For suppose ∩ M (a, b) ≠ { 0 } ⇒ ∃ a non zero element a ∈ ∩ M (a, b)

⇒ a ∈ M (a, 0)

⇒ a – 0 ∈ M (a, 0). It is a contradiction.

∴ ∩ M (a, b) = { 0 }.

Since ∩ M (a, b) = { 0 } and N / M(a, b) is subdirectly irreducible then N is a subdirect product of subdirectly irreducible near-rings N / M(a, b) .

2.3 **Definition** : An element a of a near-ring N is distributive if a (b + c) = ab + ac ∀ b , c ∈ N. An element b of a near-ring N is said to be anti-distributive if b(c + d) = bd + bc ∀ c, d ∈ N.

2.4 **Definition:** A near-ring N is said to be distributively generated if (N, +) is generated by a subset of distributive elements of N.

2.5 **Note:** It follows that an element a is distributive iff – a is anti- distributive. In particular, any element of a distributively generated near-ring is a finite sum of distributive and anti-distributive elements.

2.6 **Definition** : A near-ring N is Boolean if ∀ x ∈ N , $x^2 = x$, 2x = 0.

2.7 **Example** : N = { a_0, a_1, a_2, a_3, a_3, a_5} is an additive group (not necessarily abelian) where + is defined by the following table

+	a_0	a_1	a_2	a_3	a_3	a_5
a_0	a_0	a_1	a_2	a_3	a_3	a_5
a_1	a_1	a_0	a_5	a_3	a_3	a_2
a_2	a_2	a_3	a_3	a_0	a_5	a_1
a_3	a_3	a_5	a_0	a_2	a_1	a_3
a_3	a_3	a_2	a_1	a_5	a_0	a_3
a_5	a_5	a_3	a_3	a_1	a_2	a_0

and define • on N as follows

x • y = x ∀ x , y ∈ N. then (N , + , •) is a Boolean near-ring which is not a ring.

2.8 **Theorem** : A Boolean near-ring with left identity is zero-symmetric.

Proof. Let e, x ∈ N where e is left identity.

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$(e + x0)^2 = e + 2x0 \quad \Rightarrow \quad e + x0 = e + 2x0$

$= e$

$\therefore \quad x0 = 0 \quad \forall \ x \in N.$

$\therefore$ N is zero-symmetric near-ring.

2.9 Lemma : A Boolean near-ring N is IFP near-ring i.e $ab = 0 \Rightarrow ba = 0$ and $anb = 0 \ \forall \ n \in N$.

Proof : Every Boolean near-ring has no non-zero nilpotent elements

i.e $a^n = 0 \quad \Rightarrow \quad a = 0 \quad \forall \ a \in N.$

Let $a, b \in N$ and $ab = 0$

$(ba)^2 = (ba)(ba) = b(ab)a = b0a$

$= b0 = 0$

$\therefore \ (ba)^2 = 0 \Rightarrow ba = 0.$

Let $n \in N$. And consider anb

$(anb)^2 = (anb)(anb) = (an)ba(nb) = (an)0(nb) = 0$

$\therefore \ (anb)^2 = 0 \Rightarrow anb = 0.$

$\therefore \ ab = 0 \Rightarrow anb = 0 \ \forall \ n \in N.$

Every Boolean near-ring is an IFP near-ring.

2.10 Theorem: Let N be a distrubutively generated Boolean near-ring and let x, y , z , w be elements in N such that x, y are distributive, z is anti-distributive and w is any element in N. Then the following statements hold good.

(i) $x + x = 0$

(ii) $xy = yx$

(iii) $xz = zx$

(iv) $xw = wx$

(v) $A_x = \{ n \in N / nx = 0\}$ is an ideal of N.

(vi) If $A_x = \{0\}$, then x is an identity and (N, +) is abelian.

Proof: (i) $(x + x)^2 = (x+x)(x+x)$

$= x(x+x) + x(x+x)$

$= xx + xx + xx + xx$

$= x^2 + x^2 + x^2 + x^2$

$= x + x + x + x$

$\therefore \quad x + x = x + x + x + x$

$\Rightarrow \quad x + x = 0$

$\Rightarrow \quad -x = x$

(ii) Since x, y are distributive xy is also distributive.

$(x + y)^2 = (x+y)(x+y) = x(x+y) + y(x+y)$

$= x^2 + xy + yx + y^2 = x + xy + yx + y$

$\therefore \quad x + y = x + xy + yx + y$

$\therefore \ xy + yx = 0$

$\therefore \quad yx = -(xy) = xy$ since xy is distributive.

$\therefore \quad yx = xy$

(iii) Since z is anti distributive, - z is distributive then $x(-z) = (-z)x$.

Since x is distributive $x(-z) = -(xz)$.

Since $x(-z) = (-z)x$

$\Rightarrow - (xz) = -(zx) \quad \Rightarrow \quad xz = zx$

(iv) Since every element of distributively generated near-ring is a finite sum of distributive and anti-distributive elements then

$w = w_1 + w_2 + \ldots\ldots + w_n$, then by (ii) and (iii) we have

$$xw = x(w_1 + w_2 + \ldots\ldots + w_n) = xw_1 + xw_2 + \ldots\ldots + xw_n$$
$$= w_1x + w_2x + \ldots\ldots + w_nx = (w_1 + w_2 + \ldots\ldots + w_n)x$$
$$= wx$$

$\therefore xw = wx$

(v) $A_x = \{ n \in N / nx = 0\}$

Let $n_1, n_2 \in A_x \Rightarrow n_1x = 0 , n_2x = 0$

$\Rightarrow n_1x - n_2x = 0 \quad \Rightarrow (n_1- n_2)x = 0$

$\therefore (n_1- n_2)x \in A_x$

Let $n^1 \in N , n \in A_x$

$$(n^1 + n - n^1) x = n^1x + nx - n^1x = n^1x + 0 - n^1x$$
$$= n^1x - n^1x = 0$$

$\therefore n^1 + n - n^1 \in A_x$

$\therefore A_x$ is a normal subgroup of N.

Let $n \in A_x , n^1 \in N$,

$$(n n^1)x = n(n^1x) = n(x n^1)= (nx) n^1$$
$$= 0 n^1 = 0$$

$\therefore \quad n n^1 \in A_x$.

Let $n_1, n_2 \in N$ and $n \in A_x$

$(n_1(n_2 + n) - n_1n_2)x = n_1(n_2 + n)x – (n_1n_2)x = n_1(n_2x+ nx) – (n_1n_2)x = n_1(n_2x) - (n_1n_2)x = (n_1n_2)x - (n_1n_2)x = 0$

$\therefore n_1(n_2 + n) - n_1n_2 \in A_x$. $\therefore A_x$ is an ideal in N.

(vi) Suppose $A_x =\{0\}$

Claim : x is an identity.

Suppose x is not a right identity $\Rightarrow \exists\, y \in N \ni y \neq 0, yx \neq y$

But $(yx - y)x = 0 \Rightarrow yx – y \in A_x$.

$\Rightarrow A_x \neq \{ 0\}$, it is a contradiction. $\therefore$ x is a right identity .

$\therefore$ x is a two sided identity.

Suppose $n \in N$

$$n + n = xn + xn = (x + x)n = 0n = 0$$

$\therefore n + n = 0$.

$\therefore$ Every element of N is of order 2. $\therefore$ (N , +) is abelian.

For : Let $n, n^1 \in N$

$(n + n^1) + (n^1 + n) = 0 \Rightarrow n + n^1 = - (n^1 + n) = n^1 + n$.

$\therefore n + n^1 = n^1 + n$.

2.11 Theorem : If N is a subdirectly irreducible distributively generated Boolean near-ring then N is a Boolean near-ring with an identity.

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Proof: Suppose for every distributive element x in N, $A_x \neq \{0\}$. Since N is subdirectly irreducible $\bigcap_{x \in N} A_x = A \neq \{0\}$.
Let $w \in A$ and $w \neq 0 \Rightarrow wx = 0$ for each distributive element x in N.
Since $xw = wx = 0 \Rightarrow wz = 0$ if z is anti-distributive elemnt.
Let $y \in N$, $yw = (y_1 + y_2 + \ldots\ldots + y_n)w$
$= y_1w + y_2w + \ldots\ldots + y_nw = 0$
$\therefore A_w = N$. $w = ww = 0$, it is a contradiction.
$\therefore$ There exists a distributive element $x \ni A_x = \{0\}$.
By above 2.11 theorem x is an identity and (N, +) is abelian.
Let $n, n_1 \in N$. Since N is distributively generated
$n = w_1 + w_2 + \ldots\ldots + w_k$, $n_1 = v_1 + v_2 + \ldots\ldots + v_r$,
where w_i, v_j are distributive and anti-distributive elements.

$n\,n_1 = (w_1 + w_2 + \ldots\ldots + w_k)\,n_1$
$= w_1n_1 + w_2n_1 + \ldots\ldots + w_kn_1$
$= w_1(v_1 + v_2 + \ldots + v_r) + w_2(v_1 + v_2 + \ldots + v_r) + \ldots + w_k(v_1 + v_2 + \ldots + v_r)$
$= (w_1v_1 + w_1v_2 + \ldots + w_1v_r) + (w_2v_1 + w_2v_2 + \ldots + w_2v_r) + \ldots\ldots\ldots\ldots +$
$(w_k\,v_1 + w_kv_2 + \ldots + w_k\,v_r)$
$= (v_1w_1 + v_1w_2 + \ldots + v_1w_k) + (v_2w_1 + v_2w_2 + \ldots + v_2w_k) + \ldots\ldots\ldots\ldots +$
$(v_r\,w_1 + v_rw_2 + \ldots + v_k\,w_k)$
$= v_1n + v_2n + \ldots\ldots + v_rn = (v_1 + v_2 + \ldots\ldots + v_r)n$
$= n_1n$. $\therefore$ N is a ring i.e a Boolean ring with an identity.

2.12 Theorem : Every distributively generated Boolean near-ring N is a Boolean ring.
Proof: By known theorem, N is isomorphic to subdirect product of subdirectly irreducible near-rings N_i.
Now each N_i is a homomorphic image of N and therefore a distributively generated Boolean near-ring.
So by above 2.12 theorem, $(N_i, +)$ is abelian.
Since N is distributively generated near-ring, N is a ring.
$\therefore$ N is a Boolean near-ring.

2.13 Lemma : Let N be a Boolean near-ring. Then we have
(a) Every distributive idempotent element is central.
(b) If N has a multiplicative identity element then all elements are central.
Proof: (a). let e be a distributive element and let $x \in N$.
First we show that ex = exe
$(ex - exe)\,ex = (exe\text{-}exe)\,x = 0x = 0$
$\therefore (ex - exe)\,ex = 0$
$\Rightarrow ex\,(ex - exe) = 0$ (since $ab = 0 \Rightarrow ba = 0$) ---------- I
$(ex - exe)\,e = 0$
$\Rightarrow e\,(ex - exe) = 0$ (since $ab = 0 \Rightarrow ba = 0$) ---------- II
$(ex - exe)^2 = (ex - exe)\,(ex - exe)$
$= ex\,(ex - exe) + (-exe)\,(ex - exe)$
$= 0$ from I and II.

$\therefore$ $(ex - exe)^2 = 0 \Rightarrow ex - exe = 0$

$\Rightarrow$ ex = exe --------- III

e (xe – exe) = (exe – e^2xe) (since e is distributive)

= exe – exe = 0

$\therefore$ e (xe – exe) = 0 $\Rightarrow$ (xe – exe) e = 0

$\Rightarrow$ (xe – exe) = 0 $\therefore$ xe = exe --------- IV

From III and IV, xe = ex. $\therefore$ e is central.

(b) Suppose 1 is multiplicative identity.

Let e $\in$ N and x $\in$ N.

(1 - e) e = 0 $\Rightarrow$ e (1-e) = 0

(1 - e) xe = xe – exe

$(xe - exe)^2$ = (xe - exe) (xe - exe) = (x - ex) e (1-e)xe

= (x – ex)0xe = 0

$\therefore (xe - exe)^2$ = 0 $\Rightarrow$ xe = exe

Already ex = exe (from III)

$\therefore$ ex = xe

$\therefore$ ex = xe $\forall$ e, x $\in$ N

$\therefore$ All elements of N are central i.e Z (N) = N.

2.14 Theorem: A non-trivial Boolean near-ring N contains a family of completely prime ideals with trivial intersection.

Proof: Since N has no non-zero nilpotent elements, then N has multiplicative sub semi-groups which do not contain zero element. By Zorn's lemma, let M be any maximal multiplicative sub semi-group which do not contain zero element.

Define A(M) = {x $\in$ N/ xa = 0 for atleast a $\in$ N}

Claim : A(M) is a prime ideal.

First we show that A(M) is normal subgroup of (N, +).

Let u, v $\in$ A(M)

$\Rightarrow \exists$ a, b $\in$ M Э ua = 0, vb = 0

$\Rightarrow$ uab = 0, vab = 0 (by IFP) $\Rightarrow$ (u - v)ab = 0

$\Rightarrow$ (u - v) $\in$ A(M)

Let u $\in$ A(M) and x $\in$ N then

since u $\in$ A(M) $\Rightarrow \exists$ a $\in$ M Э ua = 0

$\Rightarrow$ (x + u - x)a = xa + ua – xa = 0 (since ua = 0)

$\Rightarrow$ (x + u – x) $\in$ A(M) $\therefore$ A(M) is a normal subgroup of (N, +).

Let x $\in$ N and u $\in$ A(M)

Since u $\in$ A(M) $\exists$ a $\in$ M Э ua = 0

$\Rightarrow$ uxa = 0 (by IFP) $\Rightarrow$ ux $\in$ A(M)

Let x, y $\in$ N and u $\in$ A(M)

Since u $\in$ A(M) $\exists$ a $\in$ M Э ua = 0

Consider [y(x+u) – yx] a = y (x+u) a – yxa

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= y(xa + ua) – yxa = yxa – yxa

= 0

∴ [y(x+u) – yx] ∈ A(M) ∴ A(M) is an ideal.

Suppose x ∉ M. Then the multiplicative sub semi-group generated by M and x must contain zero. Since N has no non-zero nilpotent elements, some finite product containing x has atleast one factor and having atleast one factor from M must be zero. Repeated application of IFP, there exist an m ∈ M such that xm is nilpotent so xm = 0

⇒ the set theoretical compliment of A(M) is M.

∴ A(M) is a completely prime ideal.

And clearly every non-zero element of M is excluded from atleast one of the prime ideals of A(M).

∴ N contains a family of completely prime ideals with trivial intersection.

2.15 Theorem : Let N be a non-trivial Boolean near-ring with identity and every non trivial homomorphic image of N contains a non-zero central idempotent, then the additive group of N is commutative.

2.16 Theorem : A distributively generated Boolean near-ring N is a commutative ring.

Proof: Suppose a is a distributive element in N by the previous 2.14 lemma, a is central. By above 2.16 theorem , (N, +) is commutative. By a known theorem, N is a ring. ∴ N is a commutative ring. By a Jacobson theorem for rings ∴ N is a commutative ring.

2.17 Theorem : Suppose N is a Boolean near-ring with identity, then N is commutative ring.

Proof: Suppose N is a Boolean near-ring with identity 1.

⇒ $\bar{1}$ is non-zero central idempotent in any ring $\bar{N}$ = N/P

where P = A(M). So by above theorems (N, +) is commutative.

Every element in N is idempotent is central, so (N, ·) is commutative

2.18 Definition : If there exists a monomorphism , N to N^1 then we say that the near-ring N is embeddable in a near-ring N^1.

2.19 Theorem: Suppose N is a finite Boolean near-ring and suppose N is embeddable in a Boolean near-ring with identity then (N, +) is Boolean ring.

Proof: Suppose N is embeddable in a near-ring N^1 with identity 1 , by the above 2.18 theorem (N^1, +) is commutative.

∴ (N, +) is commutative.

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