ON PRE A*- ALBEBRAS

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Abstract: We prove that if A is a Pre A*-algebra and $x\in A$, then $M_x=\{s\in A/s\leq x\}$ is a Pre A*-algebra and the mapping $\alpha_x : A\to M_x$ defined by $\alpha_x(s)=x\wedge s$ for all $s\in A$ is a homomorphism of A onto M_x with kernel θ_x and hence obtain the result that $A/\theta_x\cong M_x$. If B is a Boolean algebra and $a\in B$, then we know that B is isomorphic to (a] × [a). Based on this result we prove similar decomposition for a Pre A*-algebra.

1. INTRODUCTION

In[2], the Equational theory of Disjoint Alternatives,around 1989, E.G.Maines introduced the concept of Ada $(A,\wedge,\vee,(\text{-})',(\text{-})_\pi,0,1,2)$, (Algebra of Disjoint Alternatives). This result differs from the definition of the Ada [3] of his later paper. While the Ada of the prior seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras, the later concept is based on C-algebras $(A,\wedge,\vee,(-)^{\sim})$ introduced by Fernando Guzman and Craig C. Squir [1]. In 1994, P.Koteswara Rao [2] first introduced the concept of A*-algebra $(A,\wedge,\vee,*,(-)^{\sim}\,(\text{-})_\pi,0,1,2)$ and studied the equivalence with Ada [4], C-algebra [1], and Ada [4] and its connection with 3-ring, Stone type representation and introduced the concept of A*-clone and the If-Then-Else structure over A*-algebra and ideal of A*-algebra. In 2000, J.Venkateswara Rao [5] introduced the concept Pre A*-algebra $(A,\wedge,\vee,(-)^{\sim})$ analogous to C-algebra as a reduct of A*- algebra.

1. Preliminaries: In this section onwards we concentrate on the algebraic structure of Pre A*-algebra, centre of Pre A*-algebra and stated some results which are also used in the later text.

1.1. Definition: An algebra $(A,\wedge,\vee,(-)^{\sim})$ where A is a non-empty set with $1,\wedge,\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim}=x,\ \forall x\in A$

(b) $x\wedge x=x,\ \forall x\in A$

(c) $x\wedge y=y\wedge x, \forall x,y\in A$

(d) $(x\wedge y)^{\sim}=x^{\sim}\vee y^{\sim}, \forall x,y\in A$

(e) $x\wedge(y\wedge z)=(x\wedge y)\wedge z, \forall x,y,z\in A$

(f) $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z), \forall x,y,z\in A$

(g) $(x\wedge y)=x\wedge(x^{\sim}\vee y), \forall x,y\in A$ is called a Pre A*-algebra

1.2. Example: 3 = {0, 1, 2} with operations $\wedge,\vee,(-)^{\sim}$ defined below is a Pre A*-algebra.

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$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

1.3.Note:The elements 0,1,2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$

(b) $1 \wedge x = x$, $\forall\ x \in 3$

(c) $0 \vee x = x$, $\forall\ x \in 3$

(d) $2 \wedge x = 2 \vee x = 2$, $\forall\ x \in 3$.

(f) If there is an element x in A such that $x^{\sim} = x$, then it is unique and we denote it by 2. (2 is called uncertain element of A)

1.4. Example: 2 = {0, 1} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2

x	$x^{\sim}$
0	1
1	0

1.5. Note: (i) $(2, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A * algebra (ii) The identities 1.1(a) and 1.1(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.1

1.6. Lemma: [6] Every Pre A*-algebra with 1 satisfies the following laws

(a) $x \vee 1 = x \vee x^{\sim}$ (b) $x \wedge 0 = x \wedge x^{\sim}$ (c) $x \wedge (x^{\sim} \vee x) = x \vee (x^{\sim} \wedge x) = x$ (d) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$ (e) $(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$ (f)If $x \vee y = 0$ then $x = y = 0$ (g) If $x \vee y = 1$ then $x \vee x^{\sim} = 1$

1.7 Definition: Let A be a Pre A*-algebra. An element $x \in A$ is called central element of A if $x \vee x^{\sim} = 1$ and the set $\{ x \in A / x \vee x^{\sim} = 1 \}$ of all central elements of A is called the centre of A and it is denoted by $B(A)$.

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1.8. Theorem: [6] Let A be a Pre A*-algebra with 1, then $B(A)$ is a Boolean algebra with the induced operations $\wedge,\vee,(-)^{\sim}$.

1.9. Theorem: [6] Let A is a Pre A*-algebra with 1. Then A has trivial centre if and only if $A=\overline{A_o}$, for some Pre A*-algebra A_o .

1.10.Lemma: [6] Let A be a Pre A*-algebra with 1.

a) If $y\in B(A)$ then $x\wedge x^{\sim}\wedge y=x\wedge x^{\sim}$ $\forall x\in A$

(b) $x\wedge(x\vee y)=x\vee(x\wedge y)=x$ if and only if $x,y\in B(A)$

1.11. Definition: Let $(A_1,\vee,\wedge,(-)^{\sim})$ and $(A_2,\vee,\wedge,(-)^{\sim})$ be a two Pre A*- algebras. A mapping $f:A_1\to A_2$ is called an Pre A* homomorphism if (i) $f(a\wedge b)=f(a)\wedge f(b)$ (ii) $f(a\vee b)=f(a)\vee f(b)$ (iii) $f(a^{\sim})=(f(a))^{\sim}$.

1.12. Definition: Let A_1,A_2 be two Pre A*-algebras and $f:A_1\to A_2$ be a homomorphism then the set $\{x\in A_1 / f(x)=0\}$ is called the Kernal of f and it is denoted by Kerf.

1.13. Lemma: [5]. Let A be a Pre A*-algebra with 1, 0. Suppose that for every $x\in A-\{0,1\}$, $x\vee x^{\sim}\neq 1$. Define $f:A\to\{0,1,2\}$ by $f(1)=1$, $f(0)=0$ and $f(x)=2$ if $x\neq 0,1$. Then f is a Pre A*- homomorphism.

1.14. Definition: A relation θ on a Pre A*- algebra $(A,\wedge,\vee,(-)^{\sim})$ is called congruence relation if

(i) θ is an equivalence relation (ii) θ is closed under $\wedge,\vee,(-)^{\sim}$

1.15. Lemma: [5] Let $(A,\wedge,\vee,(-)^{\sim})$ be a Pre A*-algebra and let $a\in A$ then the relation $\theta_a=\{(x,y)\in A\times A / a\wedge x=a\wedge y\}$ is(a) a congruence relation(b) $\theta_a\cap\theta_{a^{\sim}}=\theta_{a\vee a^{\sim}}$ (c) $\theta_a\cap\theta_b\subseteq\theta_{a\vee b}$ (d $\theta_a\cap\theta_{a^{\sim}}\subseteq\theta_{a\wedge a^{\sim}}$

1.16.Lemma: Let $(A,\wedge,\vee,(-)^{\sim})$ be a Pre A*-algebra and let $a\in A$ then the relation $\beta_a=\{(x,y)\in A\times A / a\vee x=a\vee y\}$ is (a) congruence relation (b) $\beta_a\cap\beta_{a^{\sim}}\subseteq\beta_{a\vee a^{\sim}}$ (c) $\beta_a\cap\beta_{a^{\sim}}=\beta_{a\wedge a^{\sim}}$ (d) $\beta_a\cap\beta_b\subseteq\beta_{a\wedge b}$

1.17. Definition: [6] Let A be a Pre A*-algebra .Define a relation '$\leq$ 'on A by $x\leq y$ if and only if $y\wedge x=x\wedge y=x$

1.18. Lemma: [6] A is a Pre A*- algebra , then $(A,\leq)$ is a poset.

1.19.Definition: Let A be a Pre A*-algebra .Define '$\leq$' on A by $x \leq y$ if and only if $y \vee x = x \vee y = y$

1.20. Lemma: [6] If A is a Pre A*-algebra , $(A,\leq)$ then is a poset.

2. THE PRE A*-ALGEBRA M_x

Recall that for every Boolean algebra B and $a \in$B the set (a] = { $x \in B / x \leq a$ } ([a) = { $x \in B / a \leq x$ is a Boolean algebra under the induced operations $\wedge, \vee$ where the complementation is defined by $x^* = a \wedge x'$ ($x^* = a \vee x'$). In a similar way in this section, we prove that if A is a pre A*-algebra and $x \in A$, then $M_x = \{s \in A / s \leq x\}$ is a Pre A*-algebra under the induced operations $\wedge, \vee$ where the complementation is defined by $s^* = x \wedge s^\sim$ the relation defined on Pre A* algebra A by $s \leq x$ if $s \wedge x = x \wedge s = s$ and the mapping $\alpha_x : A \to M_x$ defined by $\alpha_x(s) = x \wedge s$ for all $s \in A$ is a homomorphism of A onto M_x with kernel θ_x and hence $A/\theta_x \cong M_x$.

2.1. Theorem: Let A be a Pre A* algebra, $x \in A$ and $M_x = \{s \in A / s \leq x\}$ .Then $< M_x, \wedge, \vee, * >$ is Pre A* algebra with 1 where $\wedge, \vee$ are the operations in A restricted to M_x, s^* is defined by $x \wedge s^\sim$, the relation defined on Pre A* algebra by $s \leq x$ if $s \wedge x = x \wedge s = s$

Proof: If $s \in M_x$,then $x \wedge s^* = x \wedge (x \wedge s^\sim) = (x \wedge x) \wedge s^\sim = x \wedge s^\sim = s^*$ .So that $s^* \in M_x$ and $s^{**} = (s^*)^* = (x \wedge s^\sim)^* = x \wedge (x \wedge s^\sim)^\sim = x \wedge (x^\sim \vee s) = x \wedge s = s$ .Therefore $s^{**} = s$ .Now, for $s,t \in M_x$, $(s \wedge t)^* = x \wedge (s \wedge t)^\sim$

$= x \wedge (s^\sim \vee t^\sim)$ $= (x \wedge s^\sim) \vee (x \wedge t^\sim) = s^* \vee t^*$ .Therefore $(s \wedge t)^* = s^* \vee t^*$

For $s,t \in M_x$, $s \wedge (s^* \vee t) = s \wedge ((x \wedge s^\sim) \vee t)$ $= s \wedge (x \wedge s^\sim) \vee (s \wedge t)$ $=$ $s \wedge (s^\sim \wedge x) \vee (s \wedge t)$ $= (s \wedge s^\sim) \vee (s \wedge t) = s \wedge (s^\sim \vee t)$ $= s \wedge t$.

Therefore $s \wedge t =$ $s \wedge (s^* \vee t)$.
The remaining properties hold in M_x since they hold in A. Hence $< M_x, \wedge, \vee, * >$ is a Pre A*- algebra.

2.2. Note:

(i) Observe that M_x is not a sub-algebra of A because the operation * is not the restriction of $\sim$ to M_x.

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(ii) $<M_x, \wedge, \vee, *>$ is a Pre A*-algebra with $s \wedge x = s \ \forall s \in A$. Hence x is an identity for $\wedge$

2.3. Theorem: Let A be a Pre A*- algebra. Then the following hold:

(i) $M_x = \{x \wedge s / s \in A\}$ (ii) $M_x = M_y \Leftrightarrow x = y$

(iii) $M_x \cap M_y = M_{x \wedge y}$ (iv) $(M_x)_{x \wedge y} = M_{x \wedge y}$

Proof: (i) It is clear from the definition. (ii) It can be proved routinely (iii) Let $s \in M_x \cap M_y$ then $s \in M_x$ and $s \in M_y$. $\Rightarrow$ $s \wedge x = s, s \wedge y = s$. Now $(x \wedge y) \wedge s = x \wedge (y \wedge s)$ $= x \wedge s = s$ .This implies that $s \in M_{x \wedge y}$ i.e., $M_x \cap M_y \subseteq M_{x \wedge y}$. Let $s \in M_{x \wedge y}$, then $(x \wedge y) \wedge s = s$, $x \wedge s = x \wedge \{(x \wedge y) \wedge s\} = x \wedge x \wedge y \wedge s = x \wedge y \wedge s = s$ .This implies $s \in M_x$ and $y \wedge s = y \wedge \{(x \wedge y) \wedge s\} = x \wedge y \wedge y \wedge s = x \wedge y \wedge s = s$. This implies $s \in M_y$ Therefore $M_x \cap M_y \supseteq M_{x \wedge y}$. Hence as required (iv) $(M_x)_{x \wedge y} = \{x \wedge y \wedge t / t \in M_x\}$ = $\{x \wedge y \wedge x \wedge s / s \in M\}$ = $\{x \wedge y \wedge s / s \in M\}$ = $M_{x \wedge y}$

2.4. Lemma: Let $f : A_1 \to A_2$ be Pre A* - homomorphism where A_1, A_2 are Pre A* algebras with $1_1$and 1_2 . Then (i) If A_1 has the uncertain element 2, then $f(2)$ is the uncertain element of A_2 (ii) If $a \in B(A_1)$, then $f(a) \in B(A_2)$

Proof: Let x be an element in A_1 then $f(x)$ is an element in A_2

(i) $f(2) = f(x \vee 2) = f(x) \vee f(2)$,also $f(2) = f(x \wedge 2) = f(x) \wedge f(2)$ and $(f(2))^{\sim} = f(2^{\sim}) = f(2)$. This shows that $f(2)$ is uncertain element in A_2 (ii) If $a \in B(A_1)$, then $a \vee a^{\sim} = 1_1$, and $f(a \vee a^{\sim}) = f(1_1) \Rightarrow f(a) \vee f(a^{\sim}) = 1_2$ .This shows that $f(a) \in B(A_2)$.

2.5. Theorem: Let A be a Pre A*-algebra with 1 and $x \in A$, then the mapping $\alpha_x : A \to M_x$ defined by $\alpha_x(s) = x \wedge s$ for all $s \in A$ is a homomorphism of A onto M_x with kernel θ_x and hence $A / \theta_x \cong M_x$

Proof: For $s \in A$, $x \wedge (x \wedge s) = x \wedge s$, $\Rightarrow$ $x \wedge s \leq x$. Hence $x \wedge s \in M_x$ .Let $s, t \in A$, $\alpha_x(s \wedge t) = x \wedge s \wedge t = x \wedge s \wedge x \wedge t = \alpha_x(s) \wedge \alpha_x(s)$ .Therefore $\alpha_x(s \wedge t) = \alpha_x(s) \wedge \alpha_x(s)$ and $\alpha_x(s \vee t) = x \wedge (s \vee t) = (x \wedge t) \vee (x \wedge s) = \alpha_x(s) \vee \alpha_x(t)$. Hence $\alpha_x(s \vee t)$ $= \alpha_x(s) \vee \alpha_x(t)$

$\alpha_x(s^{\sim}) = x \wedge s^{\sim} = x \wedge (x^{\sim} \vee s^{\sim}) = x \wedge (x \wedge s)^{\sim} = (x \wedge s)^* = (\alpha_x(s))^*$

Therefore $\alpha_x(s^{\sim}) = (\alpha_x(s))^*$. Hence α_x is a Pre A*homomorphism.

Now $s \in M_x$, we have $\alpha_x(s) = s$. Therefore α_x is onto homomorphism. Hence by the fundamental theorem of homomorphism $A / ker\ \alpha_x \cong M_x$ and

Ker $\alpha_x = \{(s,t) \in A \times A / \alpha_x(s) = \alpha_x(t)\}$ $= \{(s,t) \in A \times A / x \wedge s = x \wedge t\} = \theta_x$. Thus $A / \theta_x \cong M_x$

3. DECOMPOSITION OF A

If B is a Boolean algebra and $a \in B$, then we know that B is isomorphic to $(a] \times [a)$. In this section we prove similar decomposition for a Pre A*-algebra. First, we prove the following lemma.

3.1.Lemma: Let A be a Pre $A*$ algebra with 1, $a \in B(A)$ and $x, y \in A$, then $a \wedge x = a \wedge y,\ a^{\sim} \wedge x = a^{\sim} \wedge y \Leftrightarrow x = y$

Proof: Let $a \in B(A)$ and $x, y \in A$.

Suppose that $a \wedge x = a \wedge y$ and $a^{\sim} \wedge x = a^{\sim} \wedge y$

$x = (a \vee a^{\sim}) \wedge x = (a \vee x) \wedge (a^{\sim} \vee x) = (a \vee y) \wedge (a^{\sim} \vee y)$

$= (a \vee a^{\sim}) \wedge y = 1 \wedge y = y$ and the converse is trivial.

The above result fails if $a \notin B(A)$. For example, in Pre $A*$ - algebra A, we have $2 \notin B(A)$ and $2 \vee 1 = 2 \vee 0$ and $2^{\sim} \vee 1 = 2^{\sim} \vee 0$ but $1 \neq 0$

3.2. Theorem: If A is a Pre $A*$-algebra with 1 and $a \in B(A)$ then A can be embedded into $M_a \times M_{a^{\sim}}$

Proof: Define $\alpha : A \to M_a \times M_{a^{\sim}}$ by $\alpha(x) = (\alpha_a(x), \alpha_{a^{\sim}}(x)), \forall x \in$ A

Then, by theorem 2.5 α is well defined and α is homomorphism. If $\alpha(x) = \alpha(y) \Rightarrow$ $a \wedge x = a \wedge y$ and $a^{\sim} \wedge x = a^{\sim} \wedge y \Rightarrow x = y$. Therefore α is monomorphism. Hence as required.

In the above result, for $a = 0$, we have $M_a = \{0,2\}$, $M_{a^{\sim}} = \{0,1,2\}$ and $M_a \times M_{a^{\sim}} = \{(0,0),(0,1),(0,2),(2,0),(2,1),(2,2)\}$. So we can't find x such that $\alpha(x) = (2,0)$. Hence α is not an onto mapping. But we have the following theorem.

3.3. Theorem: If A is a Pre $A*$-algebra with 1 and $a \in B(A)$,
$M_a = \{s \in B(A) / s \leq a\}$ and $M_{a^{\sim}} = \{t \in B(A) / t \leq a^{\sim}\}$ then $B(A) \cong M_a \times M_{a^{\sim}}$

Proof: Define $\alpha : B(A) \to M_a \times M_{a^{\sim}}$ by $\alpha(x) = (\alpha_a(x), \alpha_{a^{\sim}}(x)), \forall x \in B(\mathrm{A})$

Then by the theorem 3.2, α is monomorphism.

α is onto: Let $(x, y) \in M_a \times M_{a^{\sim}}$.

Then $x \leq a$ and $y \leq a^{\sim}$. So that $a \wedge x = x$ and $a^{\sim} \wedge y = y$

Thus $a^{\sim} \wedge x = a^{\sim} \wedge a \wedge x = 0$ and $a \wedge y = a \wedge a^{\sim} \wedge x = 0$

Since $x, y \in B(A)$, $x \vee y \in B(A)$.

$\alpha(x \vee y) = (\alpha_a(x \vee y), \alpha_{a^\sim}(x \vee y)) = (a \wedge (x \vee y), a^\sim \wedge (x \vee y))$

$= ((a \wedge x) \vee (a \wedge y), (a^\sim \wedge x) \vee (a^\sim \wedge y)) = (x \vee 0, 0 \vee y) = (x, y)$.

Hence α is isomorphism.

In a Pre A*-algebra $A = \{0,1,2\}$, if $a = 2$ $(i.e, a^\sim = a, a \notin B(A))$, then $M_a = \{2\}$ and $M_{a^\sim} = \{2\}$. The mapping $\alpha : A \to M_a \times M_{a^\sim}$ defined by $\alpha(x) = (\alpha_a(x), \alpha_{a^\sim}(x)), \forall x \in A$ becomes $\alpha(x) = (\alpha_2(x), \alpha_2(x)) = (2 \wedge x, 2 \wedge x) = (2,2)$, which is constant onto homomorphism and also we can observe that, if t is an identity for $\wedge$, then the mapping $\alpha : A \to M_a \times M_t$, defined by $\alpha(x) = (\alpha_a(x), \alpha_t(x)), \forall x \in A$ becomes as $\alpha(x) = (2 \wedge x, t \wedge x), \forall x \in A = (2, x)$, $\forall x \in$ A (since t is an identity for $\wedge$)

Therefore α is well defined, one- one homomorphism. and also for each $(2, x) \in M_a \times M_t$ there exists $x \in A$ such that $\alpha(x) = (2, x), \forall x \in A$. Therefore α is onto. Hence $A \cong M_a \times M_t$.

Consider the direct product 2×3 of algebras defined in examples 1.2 and 1.4. We obtain 6- element algebra with four central elements. However, the non-central elements do not satisfy $a = a^\sim$. So we can't generalize these observations as "If A is a Pre A*-algebra, $a \notin B(A)$ the mapping $\alpha : A \to M_a \times M_{a^\sim}$ defined by $\alpha(x) = (\alpha_a(x), \alpha_{a^\sim}(x)), \forall x \in A$ is a constant onto homomorphism and $A \cong M_a \times M_t$, where t is an identity for $\wedge$".

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