ROUGH LATTICES AND ROUGH BOOLEAN ALGEBRA

B.K.Tripathy[1], M.K.Satapathy[2]

Abstract: The first attempt to extend the notion of crisp lattices to the setting of rough sets is due to Iwinski. But his definition uses the algebraic definition of rough sets and has very little relevance with Pawlak's definition of roughness. Later, rough lattices based upon Pawlak's notion of indiscernibility of elements in a set are introduced by Chakraborty, Biswas and Nanda. However, this definition of rough lattice defines the rough structure on a lattice. So, most of the properties of rough lattice involving elements are directly derivable from the order properties of the basic lattice. However, in this paper, we define a rough partial ordering relation on a set and define ordering of elements through this relation only. It seems to be the most natural one and we shall prove some properties of these lattices. Also, we define rough Boolean algebras.

Keywords: ***Rough set, partial order relation, partially ordered set, rough lattice, rough Boolean algebra.***

1. INTRODUCTION

The problem of knowledge acquisition under uncertain conditions is a major area of research. The notion of rough sets introduced by Pawlak ([10]) plays an important role in dealing with such problems. Rough sets play an important role for handling situations which are not crisp and deterministic but associated with impreciseness in the form of indiscernibility between the objects of a set. So, in case of dealing with some types of knowledge representation problems, rough algebraic structures are useful. The concepts of Lattices and Boolean algebra ([2]) are of cardinal importance in the theory and design of computers and of circuitry in general, besides having numerous other applications in mathematical logic, probability theory and other fields of engineering and mathematics. Lattice as an algebraic structure is of considerable importance, in view of its applications in different fields of physical sciences, such as quantum logic, Reynolds's operators, ergodic theory, criticality of multiplicative processes etc. ([9]). The first attempt to define rough lattices is due to Iwinski ([3, 4]). However, his definition of a rough set is algebraic by nature and does not involve the indiscernibility of elements in a set. That is why, the rough sets defined by Iwinski are not popular and very little research has been done using this approach. The next attempt to define rough lattices is due to Chakraborty et al ([8]), where the Pawlak's definition of a rough set has been used. However, this definition is developed upon a basic lattice. The rough lattice inherits the order structure of the basic lattice and all the rough lattices defined in an approximation space have the same order relation in them. This is a major drawback of this definition. Very few elementary properties of lattices are established in this work. In the present paper, we introduce a more natural definition of a rough lattice through a rough partial order relation (defined by us) and establish many of the properties of basic lattice. Also, we introduce the important concept of rough Boolean algebra and prove some of its properties.

In the next section we provide some definitions and notations to be used in the paper. In section 3, we introduce the notion of rough lattices and establish many properties. In the next section, we illustrate the concepts through some examples. In section 5, the notion of rough Boolean algebra is introduced and some properties are proved. In section 6 we summarise and mention some scope for future work.

2. DEFINITIONS AND NOTATIONS

The basic assumption in rough set theory is that human knowledge about a universe is determined by their capability to classify the objects in it. Since classifications and equivalence relations are interchangeable notions on a universe, for mathematical reasons equivalence relations are considered in defining rough sets. In the next paragraph we define rough sets and some notions closely related to it.

2.1 Rough sets

Let U be a universe of discourse and R be an equivalence relation over U, called the indiscernibility relation. By U/R we denote the family of all equivalence classes induced by R on U. These classes are referred to as categories or concepts of *R* and the equivalence class of an element $x \in U$, is denoted by $[x]_R$.

The basic concept of rough set theory is the notion of an approximation space, which is an ordered pair A = (U, R). For x, y $\in$ U, if x R y then x and y are said to be indistinguishable in A. The elements of U/R are called elementary sets in A. It is assumed that the empty set is also elementary for every approximation space. A definable set in A is any finite union of elementary sets in A.

2.1.1 Rough approximations

By a knowledge base, we understand a relational system K = (U, S), where U is as above and S is a family of equivalence relations over U. For any subset P ($\neq \phi$) $\subset$ S, the intersection of all equivalence relations in P is denoted by IND (P) and is called the indiscernibility relation over *P*. Given any $X \subseteq U$ and R$\in$ IND (K), we associate two subsets, $\underline{R}X = \cup \{Y \in U/R : Y \subseteq X\}$ and $\overline{R}X = \cup \{Y \in U/R : Y \cap X \neq \phi\}$, called the R-lower and R-upper approximations of X *respectively*.

The R-boundary of *X* is denoted by BN_R (X) and is given by $BN_R(X) = \overline{R}X \setminus \underline{R}X$. The elements of $\underline{R}X$ are those elements of U, which can certainly be classified as elements of X, and the elements of $\overline{R}X$ are those elements of U, which can possibly be classified as elements of X, employing knowledge of *R*. We say that *X* is rough with respect to *R* if and only if $\underline{R}X \neq \overline{R}X$, equivalently $BN_R(X) \neq \phi$. X is said to be R-definable if and only if $\underline{R}X = \overline{R}X$, or $BN_R(X) = \phi$.

2.1.2 Rough Membership Function

Rough sets can also be defined employing the rough membership function instead of approximation ([13]). Continuing with the notations above, we define the membership function of X with respect to R as

$\mu_X^R : X \rightarrow [0,1]$, such that

$$\mu_X^R(x) = \frac{card([x]_R \cap X)}{card([x]_R)}, \qquad (1)$$

where 'card' represents cardinality function on a set.

The value $\mu_X^R(x)$ can be interpreted as the degree that x belongs to X in view of knowledge about x expressed by R or the degree to which the elementary granule $[x]_R$ is included in the set X. This means that the definition reflects a subjective knowledge about elements of the universe, in contrast to the classical definition of a set.

The rough membership function can also be interpreted as the conditional probability, and can be interpreted as a degree of certainty to which x belongs to X.

The rough membership function can be used to define the lower approximation, the upper approximation and the boundary region of a set, as follows:

$\underline{R}(X) = \{x \in U : \mu_X^R(x) = 1\}$,

$\overline{R}(X) = \{x \in U : \mu_X^R(x) > 0\}$, and $BN_R(X) = \{x \in U : 0 < \mu_X^R(x) < 1\}$.

Also, the following two properties hold for rough membership functions.

$$\mu_{A \cup B}^R(x) \geq \max(\mu_A^R(x), \mu_B^R(x)) \text{ for any } x \in U. \qquad (2)$$

$$\mu_{A \cap B}^R(x) \leq \min(\mu_A^R(x), \mu_B^R(x)) \text{ for any } x \in U. \qquad (3)$$

2.2 Rough relations

The notion of rough relation was introduced and their properties were studied by Pawlak ([11], [12]). Some of the properties of rough relations were found to be incorrect and the correct versions are proved in [6]. Stepaniuk ([5], [6]) has established some more properties of rough relations and their applications.

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Definition 2.2.1: Let $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ be two approximation spaces. The product of A_1 by A_2 is the approximation space denoted by A = (U, S), where U = $U_1 x U_2$ and the indiscernibility relation $S \subseteq (U_1 \times U_2)^2$ is defined by $((x_1, y_1), (x_2, y_2)) \in S \Leftrightarrow (x_1, x_2) \in R_1$ and $(y_1, y_2) \in R_2$,

$x_1, x_2 \in U_1$ and $y_1, y_2 \in U_2$.

It can be easily seen that S is an equivalence relation on U. The elements (x_1, y_1) and (x_2, y_2) are indiscernible in S if and only if the elements x_1 and x_2 are indiscernible in R_1 and so are the elements y_1 and y_2 in R_2. This implies that the equivalence class of S containing (x, y), denoted by $[(x, y)]_S$, is equal to the Cartesian product of $[x]_{R_1}$ by $[y]_{R_2}$.

Definition 2.2.2: Let $(U_1 x U_2, R)$ be an approximation space, where U_1 and U_2 are nonempty sets and $R \subseteq (U_1 \times U_2)^2$ be an equivalence relation. For any relation $S \subseteq U_1 \times U_2$, we define two relations L(S) and U(S) called lower and upper approximation of S respectively given by,

$L(S) = \{(x_1, x_2) \in U_1 \times U_2 : [(x_1, x_2)]_R \subseteq S\}$,

$U(S) = \{(x_1, x_2) \in U_1 \times U_2 : [(x_1, x_2)]_R \cap S \neq \phi\}$,

where [($[(x_1, x_2)]_R$] denotes the equivalence class of relation R containing the pair (x_1, x_2).

Rough relation of S is defined as the pair (L(S), U(S)).

Definition 2.2.3: Let R be an equivalence relation defined over a set U. Consider the relation R_g on $U \times U$ defined by ‘ $(x, y) R_g (p, q)$ ‘ if and only if x R p and y R q. The relation R_g is called the second generation of R or in short, a generated relation of R.

It is easy to see that R_g is an equivalence relation in $U \times U$.

Definition 2.2.4: Let S = (U, R) be an approximation space. If R_g is the generated relation of R, we say that $S_g = (U \times U, R_g)$ is a generated approximation space of S.

Definition 2.2.5: Let S = (U, R) be an approximation space and $S_g = (U \times U, R_g)$ be its generated approximation space. Suppose, U_1, U_2 are two non-null subsets of U and T is a relation from U_1 to U_2.The rough set $R_g(T) = (\underline{R_g}(T), \overline{R_g}(T))$ of T in the generated approximation space S_g is called the rough relation of T from U_1 to U_2 and is denoted by $R_g(T)(U_1 \to U_2)$.

For every $u_1 \in U_1$ and for every $u_2 \in U_2$, we say that u_1 is roughly related to u_2 denoted by $u_1 R_g(T) u_2$ having rough membership values $r_T(u_1, u_2)$ given by

$$r_T(u_1, u_2) = \frac{card[[(u_1, u_2)]_{R_g} \cap T]}{card[(u_1, u_2)]_{R_g}}.$$

If $r_T(u_1, u_2) = 0$, we say that u_1 is roughly T – related to u_2 with rough strength nil. In other words, u_1 is not at all roughly T – related to u_2. If $r_T(u_1, u_2) = 1$, we say that u_1 is certainly T – related to u_2. If $0 < r_T(u_1, u_2) < 1$, we say that u_1 is possibly T - related to u_2 with rough strength $r_T(u_1, u_2)$.

Let S = (U, R) be an approximation space and $S_g = (U \times U, R_g)$ be its generated space. Consider a non-null subset M of U and a relation T on M.

Definition 2.2.6: The rough relation $R_g(T)(M \to M)$ is said to be

Reflexive: if and only if $\forall m \in M, (m, m) \in \overline{R_g}(T)$. (4)

Symmetric: if and only if
$\forall m_1, m_2 \in M, (m_1, m_2) \in \overline{R_g}(T) \Rightarrow (m_2, m_1) \in \overline{R_g}(T)$. (5)

transitive: if and only if $\forall m_1, m_2, m_3 \in M$,
(m_1, m_2) and $(m_2, m_3) \in \overline{R_g}(T) \Rightarrow (m_1, m_3) \in \overline{R_g}(T)$. (6)

Antisymmetric: if and only if $\forall m_1, m_2 \in M$,
$(m_1, m_2), (m_2, m_1) \in \overline{R_g}(T) \Rightarrow [m_1]_R = [m_2]_R$. (7)

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Note 2.2.1: (a) Condition (4) is equivalent to the condition that
$\forall m \in M, r_T(m,m) > 0.$

(b) Condition (5) is equivalent to the condition that
$\forall m_1, m_2 \in M, r_T(m_1, m_2) > 0 \Rightarrow r_T(m_2, m_1) > 0.$ (8)

(c) Condition (6) is equivalent to the condition that
$\forall m_1, m_2, m_3 \in M, r_T(m_1, m_2) > 0$ and $r_T(m_2, m_3) > 0 \Rightarrow$
$r_T(m_1, m_3) > 0.$ (9)

(d) Condition (7) is equivalent to the condition that $\forall m_1, m_2 \in M,$
$r_T(m_1, m_2) > 0, r_T(m_2, m_1) > 0 \Rightarrow [m_1]_R = [m_2]_R.$ (10)

Note 2.2.2: When the relation R is the equality of elements '=' the equivalence classes of R consist of individual elements and the above concepts reduce to the corresponding crisp relations.

Definition 2.2.7: A relation T is said to be a rough tolerance relation if $R_g(T)$ is reflexive and symmetric.

Definition 2.2.8: A relation T is said to be a rough equivalence relation if $R_g(T)$ is reflexive, symmetric and transitive.

Definition 2.2.9: A relation T is said to be a rough partially ordering if $R_g(T)$ is reflexive, antisymmetric and transitive.

Definition 2.2.10: For any rough partial ordering T on a non-null subset M of U, the dominating class of an element x in M is denoted by $T_{\geq[x]}$ and is defined for every y in M as

$$T_{\geq[x]}(y) = r_T(x, y).$$

Definition 2.2.11: For any rough partial ordering T on a non-null subset M of U, the dominated class of an element x in M is denoted by $T_{\leq[x]}$ and is defined for every y in M as

$$T_{\leq[x]}(y) = r_T(y, x).$$

Definition 2.2.12: For any rough partial ordering T on a non-null subset M of U, the rough greatest lower bound of M is the rough set denoted by L(T, M) and is defined by

$$L(T, M) = \bigcap_{x \in M} T_{\leq [x]}.$$

Here, the intersection operation associates the minimum of the membership values in the constituents for each element in M.

Definition 2.2.13: For any rough partial ordering T on a non-null subset M of U, the rough upper bound of M is the rough set denoted by U(T, M) and is defined by

$$U(T, M) = \bigcap_{x \in M} T_{\geq [x]}.$$

Here the intersection operation associates the minimum of the membership values in the constituents for each element in M.

Definition 2.2.14: For any rough partial ordering T on a non-null subset M of U, the rough greatest lower bound of M is a unique element x in L(T, M) such that

$$L(T, M)(x) > 0 \text{ and } r_T(y, x) > 0,$$

for all elements in the support of L(T, M). The uniqueness of x is up to its equivalence class with respect to R.

Definition 2.2.15: For any rough partial ordering T on a non-null subset M of U, the rough least upper bound of M is a unique element x in U(T, M) such that

$$U(T, M)(x) > 0 \text{ and } r_T(x, y) > 0,$$

for all elements y in the support of U(T, M). The uniqueness of x is up to its equivalence class with respect to R.

Definition 2.2.16: A crisp subset M of U with a rough partial ordering T is said to be a rough lattice if and only if for any two element set {x, y} in M, the least upper bound (l.u.b) and the greatest lower bound (g.l.b) exist in M.

We denote the l.u.b. of {x, y} by $x \vee y$ and the g.l.b of {x, y} by $x \wedge y$.
We write that (M, T) is a rough lattice on (U, R).

3. EXAMPLES

In this section we provide some examples to illustrate the concepts introduced in the previous section.

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Example 3.1

Let A = (u, R) be an approximation space, where U = {a, b, c, d, e, f, g} and U/R = {{a, b}, {c, d}, {e, f}, {g}} as shown below
A = (U, R) =

a b	c d
e f	g

$B = (U^2, S) =$

(a, g) (b, g)	(g, a) (g, b)	(c, g) (d, g)
(e, e) (f, f) (e, f) (f, e)	(a, c) (a, d) (b, d) (b, c)	(d, a) (d, b) (c, a) (c, b)
(a, e) (b, e) (a, f) (b, f)	(e, a) (e, b) (f, a) (f, b)	(c, e) (d, e) (c, f) (d, f)
(e, c) (e, d) (f, c) (f, d)	(c, d) (d, c) (d, d), (c, c)	(b, b) (a, b) (b, a) (a, a)
(g, c) (g, d)	(e, g) (f, g)	(g, e) (g, f)
	(g, g)	

(a) Let us take two non empty subsets U_1 = {a, b, c} and U_2 = {f, g} of U. We take a subset T of U_1 x U_2 as T = {(a, g), (b, g), (c, f), (c, g)}.

Then $\underline{R_g}(T)$ = {(a, g), (b, g)} and $\overline{R_g}(T)$ = {(a, g), (b, g), (c, e), (d, e), (c, f), (d, f), (c, g), (d, g)}. $r_T(a, f) = 0$

and $r_T(c, f) = ¼$.

(b) Let us take T = {(a, b), (c, d), (e, f), (g, g)}. Then

$\overline{R_g}(T)$ = {(e, e), (e, f), (f, f), (f, e), (c, d), (d, c), (d, d), (c, c), (b, b), (a, b), (b, a), (a, a), (g, g)}.

It is easy to see that $R_g(T)$ is a rough equivalence relation.

(c) Let T = {(a, g), (a, c), (c, e), (g, e), (g, g)}. Then

$\overline{R_g}(T)$ = {(a, g), (b, g), (a, c), (a, d), (b, d), (b, c), (c, e), (d, e), (c, f), (d, f), (g, e), (g, f), (g, g)}.

So, $R_g(T)$ is antisymmetric.

(d) Let T = {(a, g), (e, f), (c, d), (a, b), (g, g)}. Then

$\overline{R_g}(T) = \{(a,g),(b,g),(e,e),(e,f),(f,f),(f,e),(c,d),(d,c),(d,d),(c,c),$

$(b,b),(a,b),(b,a),(a,a),(g,g)\}$

So, $R_g(T)$ is clearly reflexive.

$R_g(T)$ is antisymmetric as

$(e,f),(f,e) \in \overline{R_g}(T)$ but $[e]_R = [f]_R$;

$(c,d),(d,c) \in \overline{R_g}(T)$ but $[c]_R = [d]_R$;

$(a,b),(b,a) \in \overline{R_g}(T)$ but $[a]_R = [b]_R$.

It is also clearly rough transitive. So, $R_g(T)$ is a rough partially ordered relation. Also, we have

$T_{\geq[a]} = \{g,b\}$. $r_T(a,g) = 1/2$ and $r_T(a,b) = 1/4$.

$T_{\geq[b]} = \phi$. $T_{\geq[c]} = \{d\}$ and $r_T(c,d) = 1/4$.

$T_{\geq[d]} = \phi$, $T_{\geq[e]} = \{f\}$ and $r_T(e,f) = 1/4$.

$T_{\geq[f]} = \phi$, $T_{\leq[a]} = \phi$, $T_{\leq[b]} = \{a\}$ and $r_T(a,b) = 1/4$.

$T_{\geq[c]} = \phi$, $T_{\leq[d]} = \{c\}$, $T_{\leq[e]} = \phi$, $T_{\leq[f]} = \{e\}$ and $r_T(e,f) = 1/4$.

Example 3.2

Let U be the set of all the students in a class. We define a relation R on U as, x R y if and only if x and y are class mates. Thus $S_g = (U \times U, R_g)$ is the generated approximation space of S, where (x, y) R_g (p, q) if and only if each of the pair of students {x, p} and {y, q} are class mates. Let M be the set of all students who have passed in first class. We define the relation T on M given by x T y if and only if both x and y bear the same surname. Then the rough relation $R_g(T) = (\underline{R_g}(T), \overline{R_g}(T))$ is a rough tolerance relation. In fact, here

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$\underline{R_g}(T)$ = {(x, y) : x, y ∈ M; x & y have the same surname and for all α, β ∈ M such that x & α are classmates, y & β are classmates; α & β bear the same surname}.

$\overline{R_g}(T)$ = {(x, y) : x, y ∈ M; x & y may or may not bear the same surname. But, there exists atleast one (α, β) such that x & α are classmates, y & β are classmates and α & β are having the same surname}.

For all x∈ M, x and x bear same surname and are classmates also. So, (x, x)∈ $\overline{R_g}(T)$. Hence, $R_g(T)$ is rough reflexive. For all x, y∈ M, if (x, y) ∈ $\overline{R_g}(T)$ and x and y have the same surname, then y, x also bear the same surname. Also, if there exists at least one (α, β) such that x and α are classmates, y and β are classmates, such that, α and β have the same surname then (y, x) satisfies the same condition. So, (y, x) ∈ $\overline{R_g}(T)$. Hence, $R_g(T)$ is rough symmetric.

It can be seen that under the following cases $R_g(T)$ is not necessarily rough transitive. If (x, y) bear the same surname but (y, z) do not bear the same surname and there exists (α, β) with y, α and z, β being classmates such that α and β have the same surname. Then (x, z) do not bear the same surname and we may not be able find a pair satisfying the other condition for (x, z) ∈ $\overline{R_g}(T)$. The same problem occurs in the other case when neither of (x, y) and (y, z) have the same surname.

4. ROUGH LATTICES AND ROUGH BOOLEAN ALGEBRA

In this section we consider rough lattices and rough Boolean algebra which are in parallel to the fuzzy lattices and fuzzy Boolean algebra introduced in [1].

Properties (2) and (3) of rough membership establish that this notion is more general than the notion of fuzzy membership. In fact, these properties show that the rough membership for union and intersection of sets cannot be computed from the membership of their constituents as it can be done in case of fuzzy sets. Moreover, the rough membership function depends upon the available knowledge (represented by R). Besides, the rough membership function, in contrast to fuzzy membership function, has a probabilistic flavour ([13, 14]).

However, necessary and sufficient conditions for equality to hold in (2) and (3) have been obtained by Pawlak and Skowron ([13]).

We require some additional notations and definitions to state the theorems.

Definition 4.1: Let A = (U, R) be an information system and let X and Z be families of subsets of U such that $Z \subseteq X$ and $|Z| > 1$, where $|Z|$ denotes the cardinality of Z. The set

$$Bd_R(Z, X) = \bigcap_{A \in Z} BN_R(A) \cap \bigcap_{A \in X-Z} (U - BN_R(A))$$

is said to be the Z-boundary region defined by X and A.

Theorem A. ([13], Theorem 1) Let Z be a class of information systems with the universe including sets X and Y. The following conditions are equivalent:

$$\mu^R_{A \cap B}(x) = \min(\mu^R_A(x), \mu^R_B(x)) \text{ for any } x \in U \text{ and } A = (U, R) \in Z. \quad (11)$$

$$Bd_R(Y, X) = \phi, \text{for any } X \supseteq Y \supseteq \{A \cap -B, -A \cap B\} \text{ and}$$

$$A = (U, R) \in Z, \text{where } X = \{A \cap B, -A \cap B, A \cap -B, -A \cap -B\}. \quad (12)$$

Theorem B. ([13], Theorem 2) Let Z be a class of information systems with the universe including sets X and Y. Then (12) is equivalent to $\mu^R_{A \cup B}(x) = \max(\mu^R_A(x), \mu^R_B(x))$ for any $x \in U$ and $A = (U, R) \in Z$. (13)

Corollary 4.1 Condition (12) is necessary and sufficient for (11) and (13) to hold for any two rough sets A and B defined over an information system A = (U, R).

Throughout this section we assume that condition (12) is satisfied for any information system we are considering.

4.1 Properties of rough lattices

For brevity of notation we shall write x = y to mean that $[x]_R = [y]_R$. We first prove the following Lemma:

Lemma 4.1.1: Let (M, T) be a rough lattice on the approximation space (U, R). Then for any two elements a, b$\in$ M, $r_T(a, b) > 0 \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$.

Proof: $a \wedge b = a \Rightarrow a \in M(T, \{a, b\}) \Rightarrow a \in T_{\leq[b]} \Rightarrow r_T(a, b) > 0.$ Again

$$r_T(a, b) > 0 \Rightarrow a \in T_{\leq[b]} \Rightarrow a \in L(T, \{a, b\}), \text{ since } a \in T_{\leq[a]}.$$

This combined with $r_T(a, b) > 0$ implies that a is the g.l.b of {a, b} or $a \wedge b = a$, the other part can be proved similarly, which completes the proof.

We state the following theorem without proof.

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Theorem 4.1.1: Let (M, T) be a rough lattice on the approximation space (U, R). Then for all $a, b, c \in M$,

$$a \wedge a = a \text{ and } a \vee a = a. \quad (14)$$

$$a \wedge b = b \wedge a \text{ and } a \vee b = b \vee a. \quad (15)$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \text{ and } (a \vee b) \vee c = a \vee (b \vee c). \quad (16)$$

$$a \wedge (a \vee b) = a \text{ and } a \vee (a \wedge b) = a. \quad (17)$$

Theorem 4.1.2: Let (M, T) be a rough lattice on the approximation space (U, R). Then for all $a, b, c \in M$,

$$r_T(a,b) > 0 \Rightarrow r_T(a \wedge c, b \wedge c) > 0 \text{ and } r_T(a \vee c, b \vee c) > 0.$$

Proof: We have by the Lemma and properties of $'\wedge'$,

$$(a \wedge c) \wedge (b \wedge c) = a \wedge (c \wedge b) \wedge c = a \wedge (b \wedge c) \wedge c = (a \wedge b) \wedge (c \wedge c)$$

$$= (a \wedge b) \wedge c = a \wedge c.$$

So, again by the Lemma $r_T(a \wedge c, b \wedge c) > 0$.

Similarly, the other part can be proved.

Proofs of the following results are similar to those in the fuzzy case and so we only state these results below.

Theorem 4.1.3: Let (M, T) be a rough lattice on the approximation space (U, R). Then for all $a, b, c \in M$,

$$r_T(a,b) > 0 \text{ and } r_T(a,c) > 0 \Rightarrow$$
$$r_T(a, b \vee c) > 0 \text{ and } r_T(a, b \wedge c) > 0. \quad (18)$$

$$r_T(b,a) > 0 \text{ and } r_T(c,a) > 0 \Rightarrow r_T(b \vee c, a) > 0 \text{ and}$$
$$r_T(b \wedge c, a) > 0. \quad (19)$$

Theorem 4.1.4: Let (M, T) be a rough lattice on the approximation space (U, R). Then for all $a, b, c \in M$,

$$r_T((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) > 0. \quad (20)$$

$$r_T((a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) > 0. \quad (21)$$

Theorem 4.1.5: Let (M, T) be a rough lattice on the approximation space (U, R). Then for all $a, b, c \in M$,

$$r_T(a,c) > 0 \Leftrightarrow r_T(a \vee (b \wedge c),(a \vee b) \wedge c) > 0. \qquad (22)$$

Definition 4.1.1: A rough lattice (M, T) on the approximation space (U, R) is said to be complete if every subset of M has a l.u.b and a g.l.b.

Definition 4.1.2: A rough lattice (M, T) on the approximation space (U, R) is said to be bounded if $\exists$ two elements $0,1 \in M$ such that $r_T(0,x) > 0$ and $r_T(x,1) > 0$ for all $x \in M$.

Definition 4.1.3: A rough lattice (M, T) on the approximation space (U, R) is said to be distributive if and only if for all $a,b,c \in M$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \qquad (23)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \qquad (24)$$

Theorem 4.1.6: In a rough lattice (M, T) on the approximation space (U, R) the statements (23) and (24) are equivalent.

Theorem 4.1.7: In a rough lattice (M, T) on the approximation space (U, R) the joint cancellation laws hold, that is

$$a \vee b = a \vee c \text{ and } a \wedge b = a \wedge c \Rightarrow b = c. \qquad (25)$$

Theorem 4.1.8: In a rough distributive lattice (M, T) on the approximation space (U, R)

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a) \text{ holds for all } a,b,c \in L. \qquad (26)$$

Definition 4.1.4: A rough lattice (M, T) on the approximation space (U, R) is said to be modular if

$$a \vee (b \wedge c) = (a \vee b) \wedge c, \text{ whenever } r_T(a,c) > 0 \text{ for all } a,b,c \in L. \qquad (27)$$

Definition 4.1.5: (M, T) be a rough lattice on the approximation space (U, R) with the lower and upper bounds 0 and 1 respectively. An element $a' \in L$ is said to be a complement of an element $a \in L$ if and only if $a \wedge a' = 0$ and $a \vee a' = 1$.

Definition 4.1.6: A bounded rough lattice (M, T) on the approximation space (U, R) is said to be complemented if every element $a \in L$ has a complement in L.

Theorem 4.1.9: Every distributive rough lattice is modular.

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Theorem 4.1.10: In a rough distributive lattice (M, T) on the approximation space (U, R) the complement of an element is unique up to equivalence.

Theorem 4.1.11: In a rough distributive lattice (M, T) on the approximation space (U, R) the two DeMorgan's laws hold true. That is,

$$(a \vee b)' = a' \wedge b' \text{ and } (a \wedge b)' = a' \vee b' \text{ for all } a, b \in L. \quad (28)$$

Note 4.1.1: The rough pentagonal lattice and the rough diamond lattice can be defined as in [1] for fuzzy pentagonal and fuzzy diamond lattices.

Definition 4.1.7: A rough chain is a partially ordered rough set (M, T) on the approximation space (U, R) in which for two elements $a, b \in L$, either $r_T(a,b) > 0$ or $r_T(b,a) > 0$.

Theorem 4.1.12: Every rough chain is a distributive rough lattice.

Theorem 4.1.12: In a complemented distributive rough lattice (M, T) on the approximation space (U, R), $\forall a, b \in L$,

$$r_T(a, b) > 0 \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow r_T(b', a') > 0. \quad (29)$$

4.2. Rough Boolean algebra

In this section we introduce the concept of rough Boolean algebra. However, to do that we require the following definitions:

Definition 4.2.1: A rough lattice (M,T) is said to be complete if every nonempty subset of M has a g.l.b and an l.u.b.

Definition 4.2.2: A rough lattice (M,T) is said to be bounded if there exists two elements 0 and 1 in M such that r(0, x) >0 and r(x, 1) > 0 for all x in M.

Definition 4.2.3: A rough lattice (M,T) is said to be distributive if and only if for all $a, b, c \in M$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad (30)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad (31)$$

Definition 4.2.4: Let (M,T) be a bounded rough lattice with the lower and upper bounds 0 and 1 respectively. An element a' in M is said to be a complement of an element a in M if and only if $a \wedge a' = 0$ and $a \vee a' = 1$.

Definition 4.2.5: A bounded rough lattice (M,T) is said to be complemented if every element a in M has a complement in M.

Definition 4.2.6: A complemented distributive rough lattice (M,T) is said to be rough Boolean algebra.

Every complemented rough lattice need to be bounded. So, every rough Boolean algebra is necessarily bounded. We denote it by $\underline{M} = (M, \wedge, \vee, 0, 1, ')$, where M is a distributive bounded rough lattice with bounds 0 and 1 and every element a in M has an unique complement denoted by a'.

Definition 4.2.7: Let $\underline{M} = (M, \wedge, \vee, 0, 1, ')$, be a rough Boolean algebra. For any two elements a and b in M, we define the operation 'ring sum' denoted by $\oplus$ as $a \oplus b = (a \wedge b') \vee (a' \wedge b)$.

Definition 4.2.8: Let $\underline{M} = (M, \wedge, \vee, 0, 1, ')$, be a rough Boolean algebra. For any two elements a and b in M, we define the operation 'ring product' denoted by $\otimes$ as $a \otimes b = a \wedge b$.

Definition 4.2.9: A complemented distributive rough lattice M with the binary operations $\oplus$ and $\otimes$ is called a Rough Boolean ring with identity 1.

We state below the extensions of some properties of Boolean algebras and Boolean rings, without proofs.

Theorem 4.2.1: In a rough Boolean algebra $\underline{M}$, for all a, b in M, $a \oplus b = 0 \Leftrightarrow [a]_R = [b]_R$.

Theorem 4.2.2: In any rough Boolean ring $(M, \oplus, \otimes)$ with identity element 1,

$$a \vee (1 \oplus a) = 1, \text{ and} \quad (32)$$

$$a \otimes (1 \oplus a) = 0, \text{ for all } a \in M \quad (33)$$

5. CONCLUSION

In this article, we defined rough lattice in the natural way; that is, through a rough partially ordering relation defined over a crisp set. Also, many other definitions related to rough lattice, which are in parallel to those for crisp lattices were introduced and their properties are defined.

We introduced another important algebraic structure, the rough Boolean algebra basing upon our definition of rough lattice. The advantage of the new approach is that the ordering of elements is generated through the rough partial ordering, instead of any pre defined ordering as done in earlier cases. Also, the indiscernibility criterion of basic rough sets has been taken care unlike the notion of rough lattice defined by Iwinski [3, 4].

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[1] *B. K. Tripathy, Senior Professor, SCSE, VIT University, Vellore – 632014, Tamilnadu, India, tripathybk@rediffmail.com*

[2] *M. K. Satapathy, Head, Department Of Mathematics, R.G. College,-765025, Padmapur, Odisha, India, msat141106@gmail.com*