IDEAL CONGRUENCES FASCINATING ON PRE A* - ALGEBRA

U.Surya Kumar[1], J.Venkateswara Rao[2], V.Ramabrahmam[3], A.Satyanarayana[4]

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***Abstract:** In this paper we introduce congruences corresponding to ideals of Pre A*-algebra A. We also prove that let I be an ideal a Pre A*-algebra A then β_I is the smallest congruence on A containing $I \times I$ and proved several results on these ideal congruences. It is proved that □(A) be the lattice of all ideals of Pre A*-algebra A then $I \rightarrow \beta_I$ is homomorphism of the lattice □(A) into the lattice Con(A) of all congruences on A. Also we characterize the factor congruences on a Pre A*-algebra A.*

***Keywords:** Pre A*-algebra, ideal, congruence, ideal congruence, factor congruence.*

1. INTRODUCTION

In 2000, J. Venkateswara Rao [5] introduced the concept of Pre A*-algebra $(A, \wedge, \vee, (\tilde{-}))$ as the variety generated by the 3-element algebra A={0,1,2} which is an algebraic form of three valued conditional logic.In [7], Satyanarayana et.al. generated Semilattice structure on Pre A*-Algebras .In [6], Venkateswara Rao.J and Srinivasa Rao.K defined a partial ordering on a Pre A*-algebra A and the properties of A as a poset are studied. In [8], Satyanarayana.A, et.al. derive necessary and sufficient conditions for pre A*-algebra A to become a Boolean algebra in terms of the partial ordering.

In this paper we discuss various properties of congruences on a Pre A*-algebra. In particular, we introduce the notion of an ideal congruences corresponding to a given ideal and prove several results on these. It is proven that if □(A) be the lattice of all ideals of Pre A*-algebra A then $I \rightarrow \beta_I$ is homomorphism of the lattice □(A) into the lattice Con(A) of all congruences on A.

1. Preliminaries:

1.1. Definition: An algebra $(A, \wedge, \vee, (\text{-})\tilde{\ })$ where A is a non-empty set with 1, $\wedge$, $\vee$ are binary operations and $(-)\tilde{\ }$ is a unary operation satisfying

(a) $x\tilde{\ }\tilde{\ } = x \quad \forall x \in A$

(b) $x \wedge x = x, \quad \forall x \in A$

(c) $x \wedge y = y \wedge x, \quad \forall x, y \in A$

(d) $(x \wedge y)\tilde{\ } = x\tilde{\ } \vee y\tilde{\ } \quad \forall x, y \in A$

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad \forall x, y, z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad \forall x, y, z \in A$

(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \quad \forall x, y \in A$ is called a Pre A*-algebra.

1.2. Example: 3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2		$\vee$	0	1	2		x	$x^{\sim}$
0	0	0	2		0	0	1	2		0	1
1	0	1	2		1	1	1	2		1	0
2	2	2	2		2	2	2	2		2	2

1.3. Note: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$

(b) $1 \wedge x = x$ for all $x \in$ **3**

(c) $0 \vee x = x$ for all $x \in$ **3**

(d) $2 \wedge x = 2 \vee x = 2$ for all $x \in$ **3**.

1.4. Example: 2 = {0, 1} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1		$\vee$	0	1		x	$x^{\sim}$
0	0	0		0	0	1		0	1
1	0	1		1	1	1		1	0

1.5 Note: (i) $\left(2, \vee, \wedge, (-)^{\sim}\right)$ is a Boolean algebra. So every Boolean algebra is a Pre A*- algebra.

(ii) The identities 1.1(a) and 1.1(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.1(b) to 1.1(g).

1.6. Definition: Let A be a Pre A*-algebra. An element $x \in A$ is called a central element of A if $x \vee x^{\sim} = 1$ and the set $\{ x \in A \,/\, x \vee x^{\sim} = 1 \}$ of all central elements of A is called the centre of A and it is denoted by B (A). Note that if A is a Pre A*-algebra with 1 then $1, 0 \in B$ (A). If the centre of a Pre A*-algebra coincides with {0, 1} then we say that A has trivial centre.

1.7. Theorem:[6] Let A be a Pre A*-algebra with 1, then B(A) is a Boolean algebra with the induced operations $\wedge$, $\vee$, $(-)^{\sim}$

1.8.Lemma: Let A be a Pre A*-algebra and $x, y \in A$. Then $x \wedge y \wedge y^{\sim} = y \wedge x \wedge x^{\sim}$.

Proof: $x \wedge y \wedge y^{\sim} = y \vee x \wedge y^{\sim}$

$= y \wedge \{x \wedge (x^{\sim} \vee y^{\sim})\}$ (By 1.1(g))

$= y \wedge \{(x^{\sim} \vee y^{\sim}) \wedge x\}$

$=\{y \wedge (x^{\sim} \vee y^{\sim})\} \wedge x$

$=(y \wedge x^{\sim}) \wedge x$

$= y \wedge x \wedge x^{\sim}$

2. CONGRUENCES ON PRE A*-ALGEBRA

2.1. Definition: Let A be a Pre A*-algebra and θ be binary relation on A. Then θ is said to be an equivalence relation on A if θ satisfies the following:

(i) Reflexive: $(x, x) \in \theta$,for all $x \in$ A

(ii)Symmetric: $(x, y) \in \theta \Rightarrow (y, x) \in \theta$,for all $x, y \in$ A

(iii)Transitive: $(x, y) \in \theta$ and $(y, z) \in \theta \Rightarrow (x, z) \in \theta$,for all $x, y, z \in$ A .

We write $x\theta$ y to indicate $(x, y) \in \theta$

2.2. Definition: A relation θ on a Pre A*- algebra $(A, \wedge, \vee, (-)^{\sim})$ is called a congruence relation if

(i) θ is an equivalence relation

(ii) θ is closed under $\wedge, \vee, (-)^{\sim}$.

2.3. Lemma: [9] Let $(A, \wedge, \vee, (-)^{\sim})$ be a Pre A*-algebra and let $a \in A$ then the relation $\theta_a = \{(x, y) \in A \times A / \ a \wedge x = a \wedge y\}$ is (a) a congruence relation

(b) $\theta_a \cap \theta_{a^{\sim}} = \theta_{a \vee a^{\sim}}$ (c) $\theta_a \cap \theta_b \subseteq \theta_{a \vee \mathrm{b}}$

(d) $\theta_a \cap \theta_{a^{\sim}} \subseteq \theta_{a \wedge a^{\sim}}$ (e) $\theta_a \wedge \theta_b = \theta_{a \wedge b}$

(f) $\theta_a \vee \theta_b = \theta_{a \vee b}$ (g) $(\theta_a)^{\sim} = \theta_{a^{\sim}}$

We will write $x\ \theta_a$ y to indicate $(x, y) \in \theta_a$.

2.4. Lemma: [9] Let $(A, \wedge, \vee, (-)^{\sim})$ be a Pre A*-algebra and let $a \in A$ then the relation $\beta_a = \{(x, y) \in A \times A / \ a \vee x = a \vee y\}$ is (a) a congruence relation

(b) $\beta_a \cap \beta_{a^{\sim}} \subseteq \beta_{a \vee a^{\sim}}$ (c) $\beta_a \cap \beta_{a^{\sim}} = \beta_{a \wedge a^{\sim}}$

(d) $\beta_a \cap \beta_b \subseteq \beta_{a \wedge \mathrm{b}}$ (e) $\beta_a \wedge \beta_b = \beta_{a \wedge b}$

(f) $\beta_a \vee \beta_b = \beta_{a \vee b}$ (g) $(\beta_a)^{\sim} = \beta_{\mathrm{a}^{\sim}}$

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We will write $x\,\beta_a\,y$ to indicate $(x, y) \in \beta_a$.

2.5. Definition: Let A be a Pre A*-algebra .Then the set of all congruences on A is denoted by Con(A). If A is Pre A*-algebra then the congruences $A \times A$ and $\{(x, x) / x \in A\}$ are denoted by ∇_A and Δ_A respectively.

2.6. Definition: Let A be a Pre A*-algebra and α, β be binary relations on A. Then we define $\alpha\, o\, \beta = \{(x, y) \in A \times A \,/\, (x, z) \in \beta$ and $(z, y) \in \alpha$ for some $z \in A\}$.

If α, β are equivalence relations then $\alpha\, o\, \beta$ need not be an equivalence relation. However if $\alpha\, o\, \beta = \beta\, o\, \alpha$ then it is known that $\alpha\, o\, \beta$ is an equivalence relation**.**

2.7. Definition: Let A be a Pre A*-algebra and $\alpha, \beta \in$ Con(A). Then α, β are said to be permutable if $\alpha\, o\, \beta = \beta\, o\, \alpha$. A subset L of Con(A) is called permutable if any two congruences in L are permutable.

2.8. Lemma: For any element a of Pre A*-algebra we define $\theta = \{(x, y) \in A \times A \,/\, a \wedge x = a \wedge y\}$ then $\theta_a = \beta_{a^{\sim}} = \{(x, y) \in A \times A \,/\, a^{\sim} \vee x = a^{\sim} \vee y\}$.

Proof: Let $a, x, y \in$ A.

Let $(x, y) \in \theta_a \Rightarrow a \wedge x = a \wedge y$

$\Rightarrow a^{\sim} \vee (a \wedge x) = a^{\sim} \vee (a \wedge y)$

$\Rightarrow a^{\sim} \vee x = a^{\sim} \vee y$ (from 1.1 Definition (g))

$\Rightarrow (x, y) \in \beta_{a^{\sim}}$

Therefore $\theta_a \subseteq \beta_{a^{\sim}}$

Let $(x, y) \in \beta_{a^{\sim}} \Rightarrow a^{\sim} \vee x = a^{\sim} \vee y$

$\Rightarrow a \wedge (a^{\sim} \vee x) = a \wedge (a^{\sim} \vee y)$

$\Rightarrow a \wedge x = a \wedge y$ (from 1.1 Definition (g))

$\Rightarrow (x, y) \in \theta_a$

Therefore $\beta_{a^{\sim}} \subseteq \theta_a$

Hence $\theta_a = \beta_{a^{\sim}}$

3. IDEAL CONGRUENCES ON PRE A*-ALGEBRA

Now we introduce the notion of the ideal congruence on a Pre A*-algebra A corresponding to an ideal I of A.

3.1. Definition: A nonempty subset I of a Pre A*-algebra A is said to be an ideal of A if the following hold

(i) a ,b $\in$ I $\Rightarrow$ a $\vee$ b $\in$ I
(ii) a $\in$ I $\Rightarrow$ x $\wedge$ a $\in$ I for each x $\in$ A

3.2.Definition: For any ideal I of a Pre A*-algebra A we define $\beta_I = \{(x, y) / a \vee x = a \vee y$ for some $a \in I\}$. That is $\beta_I = \bigcup_{a\in I} \beta_a = \bigcup_{a\in I} \theta_a$

3.3. Theorem: β_I is a congruence on a Pre A*-algebra A for any ideal I of A.

Proof: We know that the union of a class of congruences on A is again a congruence on A if the given class is directed above, in the sense that, for any two members β_1 and β_2 in that class there exist a member β in the class containing both β_1 and β_2.
Now consider C = { β_a / a $\in$ I}

Since each β_a is congruence on A, C is a class of congruence on A. Also for any a, b $\in$ I we have a $\vee$ b $\in$ I and $\beta_a \vee \beta_b = \beta_{a\vee b} \in$ C
Therefore C is a directed above class of congruences and $\bigcup_{a\in I} \beta_a$ $(=\beta_I)$ is a congruence on A.

3.4. Remark: If < x > = {a $\wedge$ x/ a $\in$ A} is the principal ideal generated by an element x in a Pre A*-algebra A, then clearly $\beta_x \subseteq \beta_{<x>}$. However equality does not hold as in the case of distributive lattices. For, consider the three element Pre A*-algebra A = {0,1,2}, < 0 > = {0, 2} and $\beta_0 = \Delta_A$ and $\beta_{<0>}$ = A$\times$A. Hence $\beta_{<0>} \not\subseteq \beta_0$.

3.5. Theorem: Let I be an ideal a Pre A*-algebra A. Then β_I is the smallest congruence on A containing I$\times$I.

Proof: We know that β_I is a congruence on a Pre A*-algebra A.

Also for any x, y $\in$ I we have x $\vee$ y $\in$ I
Now (x $\vee$ y) $\vee$ x= x $\vee$ y= (x $\vee$ y) $\vee$ y hence (x, y) $\in \beta_I$ (since x $\vee$ y $\in$ I)
Therefore I$\times$I $\subseteq \beta_I$.
Now β is any congruence on A such that I$\times$I $\subseteq \beta$.
Then (x, y) $\in \beta_I \Rightarrow$ a $\vee$ x=a $\vee$ y for some a $\in$ I
We have (x $\wedge$ x$^{\sim}$, a) $\in \beta$ (since x $\wedge$ x$^{\sim}$ $\in$ $I_0 \subseteq$ I and a $\in$ I)
$\Rightarrow$ ((x $\wedge$ x$^{\sim}$) $\vee$ x, a $\vee$ x) $\in \beta$ (since β is a congruence)
$\Rightarrow$ (x, a $\vee$ x) $\in \beta$ and for similar reason (y, a $\vee$ y) $\in \beta$
$\Rightarrow$ (x, y) $\in \beta$ (since β is transitive and a $\vee$ x=a $\vee$ y)

Therefore $\beta_I \subseteq \beta$.

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Then β_I is the smallest congruence on A containing I×I.

3.6. Theorem: For any ideals I and J of a Pre A*-algebra A the following hold.

(1) $I \subseteq J \Rightarrow \beta_I \subseteq \beta_J$

(2) $\beta_I \cap \beta_J = \beta_{I \cap J}$

(3) $\beta_I \vee \beta_J = \beta_{I \vee J}$

Proof: Let I and J are ideals of a Pre A*-algebra A.

(1) Suppose that $I \subseteq J$.

Let $a \in I \Rightarrow a \in J$

Let $(x, y) \in \beta_I \Rightarrow a \vee x = a \vee y$ for some $a \in I$

$\Rightarrow a \vee x = a \vee y$ for some $a \in J$

$\Rightarrow (x, y) \in \beta_J$

Therefore $\beta_I \subseteq \beta_J$.

(2) Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$ we get $\beta_{I \cap J} \subseteq \beta_I \cap \beta_J$

Let $(x, y) \in \beta_I \cap \beta_J \Rightarrow (x, y) \in \beta_I$ and $(x, y) \in \beta_J$

$\Rightarrow a \vee x = a \vee y$ and $b \vee x = b \vee y$,where $a \in I$, $b \in J$

Now $a \wedge b \in I \cap J$ and also $(a \wedge b) \vee x = (a \vee x) \wedge (b \vee x)$

$= (a \vee y) \wedge (b \vee y)$

$= (a \wedge b) \vee y$

Therefore $(x, y) \in \beta_{I \cap J}$

Hence $\beta_I \cap \beta_J \subseteq \beta_{I \cap J}$

Therefore $\beta_I \cap \beta_J = \beta_{I \cap J}$

(3) Since $I \subseteq I \vee J$ and $J \subseteq I \vee J$ we have $\beta_I \subseteq \beta_{I \vee J}$, $\beta_J \subseteq \beta_{I \vee J}$ and hence $\beta_I \vee \beta_J \subseteq \beta_{I \vee J}$

Let $(x, y) \in \beta_z$ where $z \in I \vee J$

$\Rightarrow z = \bigvee_{i=1}^{n} x_i$ for some $x_i \in I \vee J$ and $(x, y) \in \beta_{\bigvee_{i=1}^{n} x_i} = \bigvee_{i=1}^{n} \beta_{x_i}$

(since $\beta_{a \vee b} = \beta_a \vee \beta_b$)

$\Rightarrow (x, y) \in \bigvee_{i=1}^{n} \beta_{x_i} \subseteq \beta_I \vee \beta_J$ (since each $x_i \in I$ or J)

$\Rightarrow (x, y) \in \beta_I \vee \beta_J$

$\Rightarrow \beta_{I \vee J} \subseteq \beta_I \vee \beta_J$

Therefore $\beta_I \vee \beta_J = \beta_{I \vee J}$.

Let us recall that the set Con(A) of all congruences on any algebra A is an algebraic lattice under the inclusion ordering in which the g.l.b and l.u.b of any subset Č of Con(A) are given by g.l.b Č = $\underset{\theta \in Č}{\theta}$ and l.u.b Č = $\bigcup \{ \theta_1 \, o \, \theta_2 \, o \ldots .. o \, \theta_n / \theta_i \in$ Č }. Also from [10] it is known that the set □(A) of all ideals of a Pre A*-algebra A forms an algebraic lattice under the inclusion of ordering. Now we have the following.

3.7. Theorem: Let □(A) be the lattice of all ideals of Pre A*-algebra A. Then I $\rightarrow \beta_I$ is homomorphism of the lattice □(A) into the lattice Con(A) of all congruences on A.

Proof: From 3.5 Theorem it follows that I $\rightarrow \beta_I$ is lattice homomorphism of □(A) into the lattice Con(A).

The above map I $\rightarrow \beta_I$ need not be an injection, in general. However, we have the following.

3.8. Theorem: For any Pre A*-algebra A, the map I $\rightarrow \beta_I$ of □(A) into Con(A) is an injective if A is a Boolean algebra.

Proof: Suppose that A is a Boolean algebra and I, J are ideals of A such that
$\beta_I = \beta_J$.
Then for any a$\in$ I and b$\in$ J, we have a $\vee$ (a$\vee$b) = a$\vee$b
$\Rightarrow$ (a$\vee$b, b) $\in \beta_a \subseteq \beta_I = \beta_J$

and hence x$\vee$ (a$\vee$b) = x$\vee$b, for some x$\in$ J which implies that

a = a$\wedge$ (x$\vee$ (a$\vee$b)) (since A is a Boolean algebra)
= a$\wedge$ (x $\vee$b)
$\in$ J (Since x$\vee$b$\in$ J, J is an ideal)

Therefore I$\subseteq$J and similarly J$\subseteq$I and hence I = J
Thus I $\rightarrow \beta_I$ is an injective.

3.9. Definition: Let A be a Pre A*-algebra and $\alpha \in$ Con(A). Then α is called a factor congruence if there exist $\beta \in$ Con(A) such that $\alpha \cap \beta = \Delta_A$ and $\alpha \, o$ β = A$\times$A. In this case β is called direct complement of α.

3.10. Note: In the following theorem we consider Pre A*-algebra A induced by a Boolean algebra.

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3.11. Theorem: Let A be a Pre A*-algebra with 1, θ is a factor congruence on A and β a direct complement of θ. Then there exist unique $a \in$ A such that $\theta = \theta_a$ and $\beta = \theta_{a^{\sim}}$ $(= \beta_a)$.

Proof: Let 1$^{\sim}$= 0.Then 1 and 0 are identities for operators $\wedge$and $\vee$ respectively in A.

We have $\theta \cap \beta = \Delta_A$ and $\theta \; o \; \beta =$ A$\times$A.

Then clearly $\theta \; o \; \beta = \beta \; o \; \theta =$ A$\times$A.

Since (0,1) $\in$ A$\times$A=$\theta \; o \; \beta$, there exist a$\in$ A such that (0,a) $\in \beta$ and (a,1) $\in \theta$.

First we observe that a is a unique element with the above property. If b$\in$ A also is such that (0, b) $\in \beta$ and (b, 1) $\in \theta$ then by the transitive and symmetry of β and θ we get that (a, b) $\in \theta \cap \beta = \Delta_A$, the diagonal of A and hence a=b

Thus a is unique such that (0, a) $\in \beta$ and (a,1) $\in \theta$

Now we prove that $\theta = \theta_a$ and $\beta = \theta_{a^{\sim}}$

For any x, y$\in$ A we have

(0, a$\wedge$x)= (0$\wedge$x, a$\wedge$x) $\in \beta$ (since (0, a)$\in \beta$) and hence (a$\wedge$ x, a$\wedge$y) $\in \beta$

Now (x, y) $\in \theta \Rightarrow$ (a$\wedge$ x, a$\wedge$ y) $\in \theta \cap \beta = \Delta_A$

$\Rightarrow$ a$\wedge$ x = a$\wedge$ y

$\Rightarrow$ (x, y) $\in \theta_a$

Therefore $\theta \subseteq \theta_a$.

On the other hand for any x $\in$ A, (a$\wedge$ x, x) = (a$\wedge$x, 1$\wedge$x) $\in \theta$ (since (a, 1) $\in \theta$)

Now (x, y) $\in \theta_a \Rightarrow$ a$\wedge$ x= a$\wedge$ y

We have (a$\wedge$ x, x) $\in \theta$, (a$\wedge$y, y) $\in \theta$ and a$\wedge$ x= a$\wedge$ y $\Rightarrow$ (x, y) $\in \theta$

Therefore $\theta_a \subseteq \theta$.

Thus $\theta = \theta_a$.

Also from (0, a) $\in \beta$ and (a,1) $\in \theta$ we have that (0, a$^{\sim}$) $\in \theta$ and (a$^{\sim}$,1) $\in \beta$ and by interchanging θ and β in the above argument we get that

$$\beta = \theta_{a^{\sim}} = \beta_a = \{(x, y) \in A \times A / \; a \vee x = a \vee y\}$$

we have already proved that a is unique with this property.

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[1]*Author 1 U.Surya kumar, Senior Lecturer in Mathematics, Akkineni Nageswara Rao College Gudiwada-521301, Andra Pradesh, India; uskmat1963@yahoo.in*

[2]*Author 2 : Dr. J. Venkateswara Rao Professor, Mekelle University, Mekelle, Ethiopia, venkatjonnalagadda@yahoo.co.in*

[3]*Author 3 : V. Ramabrahmam Senior Lecturer in Mathematics,Sir CRR College Eluru-534007, Andra Pradesh, India; rbveduruvada@gmail.com*

[4]*Author 4: A. Satyanarayana, Senior Lecturer in Mathematics, Akkineni Nageswara Rao College Gudiwada-521301, Andra Pradesh, India; asnmat1969@yahoo.in*