MORITA CONTEXTS FOR REDUCED AND RELATED CLASSES OF RINGS

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Abstract: In this paper we show that the ring T of a Morita context (R, S, M, N, τ, μ) is reduced if and only if both M and N are zero modules and both R and S are reduced. We also give a characterization of reduced modules over formal triangular rings.

Keywords: Morita context, reduced rings, reversible rings, semi-commutative rings.

1. INTRODUCTION

Morita theory plays an important role in the category of rings and modules. The class of rings of Morita contexts contains all 2×2 matrix rings and formal triangular rings. During the last few decades, Morita contexts were studied by people from various points of view (eg. [2], [4]). The aim of this paper is to exhibit necessary and sufficient conditions for the ring of a Morita context to be reduced. Similar conditions are studied for related classes of rings such as reversible, semi-commutative and abelian rings. We also establish criteria for modules over formal triangular rings to be reduced.

Let R and S be rings, M an $R-S-$bimodule and N an $S-R-$bimodule. We use the term "the *Morita context* (R,S,M,N,τ,μ)" to stipulate that τ is an $R-R-$homomor -phism of $M \otimes_S N$ to R and μ is an $S-S-$homomorphism of $N \otimes_R M$ to S satisfying (with notations $\tau(m \otimes f) = (m,f)$ and $\mu(f \otimes m) = [f,m]$) the following two conditions:

$[f,m]\,g = f\,(m,g)$, for all $g \in N$ and $x\,[f,m] = (x,f)\,m$, for all $x \in M$.
The bimodule homomorphisms τ and μ are called *pairings.* In the special case when both τ and μ are the zero homomorphisms, (R,S,M,N,τ,μ) is called a *Morita context with zero pairings*. With notations as above, let

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ f & \sigma \end{pmatrix} : a \in R, \quad m \in M, \quad f \in N, \quad \sigma \in S \right\}.$$

Then $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is a ring under componentwise addition and multiplication defined by

$$\begin{pmatrix} a & m \\ f & \sigma \end{pmatrix} \begin{pmatrix} b & n \\ g & \omega \end{pmatrix} = \begin{pmatrix} ab + \tau(m \otimes g) & an + (m)\omega \\ fb + \sigma g & \mu(f \otimes n) + \sigma\omega \end{pmatrix}$$

This ring (denoted by T) is the ring of the Morita context (R,S,M,N,τ,μ). Haghany and Varadarajan [4] studied such rings with $M = 0$ and called them formal triangular matrix rings.

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Throughout this paper, all rings are associative with identity. A ring R is *reduced* if it has no nonzero nilpotent elements; R is *reversible* if whenever elements $a, b \in R$ satisfy $ab = 0$, we have $ba = 0$. It can be shown easily that reduced rings are reversible (see [3]). All commutative rings are reversible, so a commutative non reduced ring is an example of a ring which is reversible but not reduced. R is a *semi-commutative* or a *ZI* (which stands for *zero-insertive*) ring if whenever elements $a, b \in R$ satisfy $ab = 0$, we have $aRb = 0$. R is *abelian* if each of its idempotent elements is central. It can be shown easily that reversible rings are semi-commutative. It was recorded in [5, §4] that the class of semi-commutative rings lies between the classes of reduced rings and abelian rings. In fact, Lambek [6, Proposition 1.3] showed that reversible rings are necessarily semi-commutative.

The notations $Nil\,(R)$ and $I\,(R)$ shall stand for the set of all nilpotent elements and idempotent elements of R respectively.

We begin our study with the case of rings of Morita contexts with zero pairings.

Lemma 1. *Let T be the ring of a Morita context (R, S, M, N, τ, μ) with zero pairings. Then*

$$Nil\,(T) = \begin{pmatrix} Nil\,(R) & M \\ N & Nil\,(S) \end{pmatrix}.$$

Proof. If $\begin{pmatrix} a & m \\ f & \sigma \end{pmatrix} \in Nil\,(T)$, then $\begin{pmatrix} a & m \\ f & \sigma \end{pmatrix}^p = 0$, for some $p \geq 1$. Now $\begin{pmatrix} a & m \\ f & \sigma \end{pmatrix}^p = \begin{pmatrix} a^p & m_1 \\ f_1 & \sigma^p \end{pmatrix}$, for some $m_1 \in M$, $f_1 \in N$. So $a \in Nil\,(R)$ and $b \in Nil\,(S)$. Conversely, if $a^s = 0$ and $b^t = 0$, for some $s, t \geq 1$, then

$$\begin{pmatrix} a & m \\ f & \sigma \end{pmatrix}^{2(s+t)} = \begin{pmatrix} a & m \\ f & \sigma \end{pmatrix}^{s+t} \begin{pmatrix} a & m \\ f & \sigma \end{pmatrix}^{s+t}$$
$$= \begin{pmatrix} a^{s+t} & m_2 \\ f_2 & \sigma^{s+t} \end{pmatrix} \begin{pmatrix} a^{s+t} & m_2 \\ f_2 & \sigma^{s+t} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & m_2 \\ f_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & m_2 \\ f_2 & 0 \end{pmatrix} = 0,$$

for some $m_2 \in M$, $f_2 \in N$. So $\begin{pmatrix} a & m \\ f & \sigma \end{pmatrix} \in Nil\,(T)$. □

Proposition 2. Let T be the ring of a Morita context (R, S, M, N, τ, μ) with zero pairings. If T is reduced, then both R and S are reduced.

Proof. Follows from Lemma 1. □

Proposition 3. *Let T be the ring of a Morita context (R, S, M, N, τ, μ) with zero pairings. Then*

$$T/Nil\,(T) \cong R/Nil(R) \times S/Nil(S).$$

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Proof. The map $\theta: T \rightarrow \begin{pmatrix} R/Nil(R) & 0 \\ 0 & S/Nil(S) \end{pmatrix}$ given by

$$\theta\begin{pmatrix} a & m \\ f & \sigma \end{pmatrix} = \begin{pmatrix} a + Nil(R) & 0 \\ 0 & \sigma + Nil(S) \end{pmatrix}$$

for any $\begin{pmatrix} a & m \\ f & \sigma \end{pmatrix} \in T$ is an onto ring homomorphism whose kernel is $Nil\ (T)$□

Proposition 3 above shows that although R and S are reduced, the ring T of the Morita context (R, S, M, N, τ, μ) with zero pairings need not be reduced. However in view of Proposition 2, we get the following necessary and sufficient condition for the ring T to be reduced.

Theorem 4. *Let T be the ring of a Morita context (R, S, M, N, τ, μ) with zero pairings. Then T is reduced if and only if $M = 0, N = 0$ and both R and S are reduced.*

Proof. If T is reduced then both R and S are reduced. That M and N are both zero modules follows from the fact that if $m \in M$ and $f \in N$, then the matrices $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}$ are nilpotent elements of T. Conversely if both M and N are zero, then by Lemma 1, $Nil\ (T) = Nil\ (R) \times Nil\ (S)$, so that T is reduced if both R and S are reduced. □

It may be conjectured that if T is the ring of a Morita context (R, S, M, N, τ, μ) with zero pairings, then $I(T) = \begin{pmatrix} I\ (R) & M \\ N & I(S) \end{pmatrix}$. That this is not true in general can be seen by the following example.

Example 5. Let R be the ring of integers and $S = R = M = N$. Then $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \notin I(R)$, although $1 \in I(R) = I(S)$.

The proof of the following lemma is fairly straightforward.

Lemma 6. *Direct products of reduced (reversible, semi-commutative, abelian) rings are reduced (respectively, reversible, semi-commutative, abelian).*

Theorem 7. *Let T be the ring of a Morita context (R, S, M, N, τ, μ) (not necessarily with zero pairings). Then*

T is reduced if and only if $M = 0, N = 0$ and both R and S are reduced.
T is reversible if and only if $M = 0, N = 0$ and both R and S are reversible.
T is semi-commutative if and only if $M = 0, N = 0$ and both R and S are semi-commutative.
T is abelian if and only if $M = 0, N = 0$ and both R and S are abelian.

Proof. (i) Suppose that T is reduced. Since for any $m \in M$, the matrix $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in Nil\,(T)$, it has to be 0 and so $m = 0$. Similarly for any $f \in N$, the matrix $\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \in Nil\,(T)$. Therefore $f = 0$. Again if $a \in R$ such that $a^2 = 0$, then $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in Nil\,(T)$. So we get $a = 0$ and hence R is reduced. Similarly, it can be shown that S is reduced.

(ii) Suppose that T is reversible. Since for any $m \in M$,

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

we have $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so that $m = 0$. Similarly for any $f \in N$,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

yielding $f = 0$.

(iii) Suppose that T is semi-commutative. By looking at the matrices $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we get that $N = 0$. Similarly the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ yield $M = 0$.

(iv) Suppose that T is abelian. As $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in I(T)$, we get that both M and N are zero.

Converses to all of the above criteria follows from Lemma 6. □

A left $R-$ module M is *reduced* (see Lemma 1.2 of [7]) if whenever $a \in R$, $m \in M$ satisfy $a^2 m = 0$, we have $aRm = 0$. In what follows, T denotes the formal triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. It is well known (see [4]) that the category $Mod - T$ is equivalent to the category Ω of triples $(X, Y)_f$ where $X \in Mod - R, Y \in Mod - S$ and $f : X \otimes_R M \to Y$ is an S-homomorphism. We shall denote the right T-module $(X, Y)_f$ by $(X \oplus Y)_T$. It can be verified that $(X \oplus Y)_T$ is a right T-module via:

$$(x, y)\begin{pmatrix} a & m \\ 0 & \sigma \end{pmatrix} = (xa, f(x \otimes m) + y\sigma)$$

and that the set $L = \{x \in X \mid f(x \otimes m) = 0,\ \forall m \in M\}$ is an R-submodule of X. The following theorem gives a necessary and sufficient condition for the right T-module $(X, Y)_f$ to be reduced.

Theorem 8. $(X \oplus Y)_T$ *is reduced over* T *if and only if* f *is the zero homomorphism and both* X_R *and* Y_S *are reduced modules.*

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Proof. Suppose $(X \oplus Y)_T$ is reduced over T. Let $x \in X$, $a \in R$ such that $xa^2 = 0$. Then $(x,\ 0)\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}^2 = 0$. As $(X \oplus Y)_T$ is reduced, $(x,\ 0)T\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = 0$, yielding $xRa = 0$, and hence X_R is reduced. Similarly it can be shown that Y_S is reduced. Now let $x \in X$ and $m \in M$. Since $(X \oplus Y)_T$ is reduced and $(x,\ 0)\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}^2 = 0$, we have $(0,\ f(x \otimes m)) = (x,\ 0)\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = 0$. Therefore f is the zero homomorphism. Conversely, let $(x, y) \in (X \oplus Y)_T$ and $\begin{pmatrix} a & m \\ 0 & \sigma \end{pmatrix} \in T$, such that $(x, y)\begin{pmatrix} a & m \\ 0 & \sigma \end{pmatrix}^2 = 0$. Since f is the zero homomorphism, we get $(xa^2, y\sigma^2) = 0$. Again as both X_R and Y_S are reduced modules, we have $xRa = 0$ and $yS\sigma = 0$. Thus for any $\begin{pmatrix} b & n \\ 0 & \omega \end{pmatrix} \in T$, we get $(x, y)\begin{pmatrix} b & n \\ 0 & \omega \end{pmatrix}\begin{pmatrix} a & m \\ 0 & \sigma \end{pmatrix} = (xba,\ y\omega\sigma) = 0$. □

Corollary 9. Consider the following R-submodule of X
$L = \{x \in X \mid f(x \otimes m) = 0,\ \forall m \in M\}$.
Then $(L \oplus Y)_T$ *is reduced over* T *if and only if both* X_R *and* Y_S *are reduced modules*.

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3. REFERENCES

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