COEFFICIENT INEQUALITIES FOR CERTAIN SUB CLASSES OF ANALYTIC FUNCTIONS

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Abstract: The objective of this paper is to introduce a subclass of analytic function using convolution and to obtain the Coefficient Inequalities and Fekete-Szego Inequality for functions in this class.

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1. INTRODUCTION

Let A be the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

In the open unit disk $U = \{z \mid z \in C : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

For any function f of the form (1.1) and for $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the convolution or Hadamard product is denoted by $f * g$ and it is defined by $f * g = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad \forall z \in U$. Let $S^*(\beta)$ and $C(\beta)$ be the subclasses of S consisting of the functions f of the form (1.1) and satisfying the conditions,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \text{ And}$$

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad \text{Respectively,}$$

for some β $(0 \le \beta < 1)$ and $z \in$ U. These classes are known as Star like and Convex functions of order β respectively.

A function f of the form (1.1) and satisfying the condition,

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$\mathrm{Re}(f'(z)) > 0 \ \forall z \in U$ is said to be a function whose derivative has a positive real part. This class is denoted by R and is studied by Macgregor [2].

S. Shams, S.R. Kulkarni and J.M. Jahangiri [6] have introduced the classes $SD(\alpha, \beta)$ and $KD(\alpha, \beta)$.

A function $f \in A$ which is of the form (1.1) and satisfies the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \qquad (z \in U)$$ is said to be in the class $SD(\alpha, \beta)$ for some $\alpha \geq 0$ and $0 \leq \beta < 1$.

Similarly $KD(\alpha, \beta)$ be the subclass of A consisting of functions $f \in A$ which are of the form (1.1) satisfying the condition.

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha\left|\frac{zf''(z)}{f'(z)}\right| + \beta \qquad (z \in U)$$

for $0 \leq \alpha \leq \beta$. These classes have been studied by S. Owa, Y. Polatogalu and E. Yavuz [3]. They have obtained the coefficient inequalities and distortion properties for the functions in these classes.

Several authors have obtained the Fekete-Szego Inequality for the normalized analytical functions f of the form (1.1) in various subclasses of S. In this paper, we define a more generalized subclass of analytic functions and obtain, the coefficient inequalities, Fekete-Szego inequality for the functions in this class. The results of this paper will generalize several earlier results in this direction.

Definition (1.1): A function $f \in A$ which is of the form (1.1) is said to be in the class $Q(\phi, \Psi, \alpha, \beta)$ if it satisfies the condition,

$$\mathrm{Re}\left(\frac{f(z)*\phi(z)}{f(z)*\Psi(z)}\right) > \alpha\left|\frac{f(z)*\phi(z)}{f(z)*\Psi(z)} - 1\right| + \beta \qquad \forall z \in U \tag{1.2}$$

Here $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$

$$\Psi(z) = z + \sum_{n=2}^{\infty} c_n z^n, \quad 0 \leq \alpha \leq \beta < 1.$$

It is noted that for various choices of ϕ, Ψ, α and β, we get several subclasses of analytical functions as follows:

i) $Q\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \beta\right) = SD(\alpha,\beta)$

ii) $Q\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \beta\right) = KD(\alpha,\beta)$

Studied by S.Owa, Y.Polatagla, E.Yavuz[3]

iii) $Q\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, 0, \beta\right) = S^*(\beta)$

iv) $Q\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, 0, \beta\right) = C(\beta)$

Studied by Robertson[5]

v) $Q\left(\frac{z}{(1-z)^2}, z\ 0{,}0\right) = R$ Studied by Macgregor [2]

To prove our results, we require the following Lemmas.

Lemma 1.1[4]: If $P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$.. is an analytic function in U with Re($P(z)$) > 0 then for any complex number v, we have

$\left|p_2 - vp_1^2\right| \leq$ 2 max [1, |2 v – 1|] and the result is sharp for the functions $P(z) = \frac{1+z^2}{1-z^2}$ and $P(z) = \frac{1+z}{1-z}$.

Lemma1.2 [1]: If $P_1(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$.. Is an analytic function in U with Re ($P_1(z)$) > 0 then for any real number v, we have

$$\left|p_2 - vp_1^2\right| \leq -4v+2 \quad \text{if } v \leq 0$$

$$\leq 2 \quad \text{if } 0 \leq v \leq 1$$

$$\leq 4v-2 \quad \text{if } v \geq 1.$$

When $v < 0$ or $v > 1$ the equality holds if and only if $P_1(z)$ is $\frac{1+z}{1-z}$ or one of it's rotations.

If $0 < v < 1$, then the equality holds true if and only if $P_1(z)$ is $\frac{1+z^2}{1-z^2}$ or one of it's rotations.

If $v = 0$ the equality holds true if and only if

$$p_1(z) = \left(\frac{1}{2}+\frac{1}{2}\alpha\right)\left(\frac{1+z}{1-z}\right)+\left(\frac{1}{2}-\frac{1}{2}\alpha\right)\left(\frac{1-z}{1+z}\right)(0 \le \alpha \le 1)$$ or one of its rotations. If v = 1 the equality holds true if and only if $P_1(z)$ is the reciprocal of one of the functions such that the equality holds true in the case when v = 0.

Although the above upper bound is sharp, in the case when $0 < v < 1$, it can be further improved as follows

$$\left|p_2 - vp_1^2\right| + v\left|p_1^2\right| \le 2 \qquad \left(0 < v \le \frac{1}{2}\right)$$

$$\left|p_2 - vp_1^2\right| + (1-v)\left|p_1^2\right| \le 2 \qquad \left(\frac{1}{2} \le v \le 1\right)$$

2. COEFFICIENT INEQUALITIES

In this section, we obtain the coefficient inequalities for the function *f* in the class

$Q(\phi, \psi, \alpha, \beta)$.

Theorem 2.1: If $f(z) \in Q(\phi, \psi, \alpha, \beta)$ with $0 \le \alpha \le \beta$ then

$$f(z) \in Q\left(\phi, \psi, \frac{\beta-\alpha}{1-\alpha}\right).$$

Proof: Since $f(z) \in Q(\phi, \psi, \alpha, \beta)$ then by definition (1.1), we have

$$\operatorname{Re}\left(\frac{f(z)*\phi(z)}{f(z)*\Psi(z)}\right) > \alpha\left|\frac{f(z)*\phi(z)}{f(z)*\Psi(z)}-1\right|+\beta.$$

Since $|z| \ge \operatorname{Re}(z)$ for any complex number z we have

$$\operatorname{Re}\left(\frac{f(z)*\phi(z)}{f(z)*\Psi(z)}\right) > \alpha\operatorname{Re}\left(\frac{f(z)*\phi(z)}{f(z)*\Psi(z)}-1\right)+\beta$$

$$\operatorname{Re}\left(\frac{f(z)*\phi(z)}{f(z)*\Psi(z)}\right) > \frac{\beta-\alpha}{1-\alpha} \tag{2.1}$$

As $0 \le \alpha \le \beta$ implies $0 \le \dfrac{\beta-\alpha}{1-\alpha} < 1$.

Hence $f(z) \in Q\left(\phi, \psi, \dfrac{\beta-\alpha}{1-\alpha}\right)$.

Theorem 2.2: If $f(z) \in Q(\phi, \psi, \alpha, \beta)$ then

$$|a_2| \le \frac{2(1-\beta)}{(1-\alpha)\,(b_2-c_2)} \tag{2.2}$$

$$|a_n| \le \frac{2(1-\beta)}{(1-\alpha)\,(b_n-c_n)} \mathop{\pi}_{j=1}^{n-2}\left[1+\frac{2(1-\beta)}{(1-\alpha)\,(b_{j+1}-c_{j+1})}.c_{j+1}\right] \text{for } n \ge 3. \tag{2.3}$$

Proof: Since $f(z) \in Q(\phi, \psi, \alpha, \beta)$ from (2.1) we have

$$\operatorname{Re}\left(\frac{f(z)*\varphi(z)}{f(z)*\psi(z)}\right) > \frac{\beta-\alpha}{1-\alpha}$$

Define a function $P(z)$ such that

$$P(z) = \frac{(1-\alpha)\dfrac{(f(z)*\varphi(z))}{(f(z)*\psi(z))}-(\beta-\alpha)}{1-\beta} = 1+\sum_{n=1}^{\infty} p_n z^n \tag{2.4}$$

Here $P(z)$ is analytic in U with $P(0) = 1$ and $\operatorname{Re}(P(z)) > 0$

From (2.4) we get

$$(1-\beta)\left[1+\sum_{n=1}^{\infty} p_n z^n\right][f(z)*\psi(z)] = (1-\alpha)[f(z)*\varphi(z)]-(\beta-\alpha)[f(z)*\psi(z)] \tag{2.5}$$

Replacing $f(z)$, $\varphi(z)$, and $\psi(z)$ with their equivalent expressions in series on both sides of the above equation (2.5), we get

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$$(1-\beta)\left[z+\sum_{n=}^{\infty}a_nc_nz^n+\sum_{n=1}^{\infty}p_nz^{n+1}+\left[\sum_{n=1}^{\infty}p_nz^n\right]\left[\sum_{n=2}^{\infty}a_nc_nz^n\right]\right]$$

$$=(1-\alpha)\left[z+\sum_{n=2}^{\infty}a_nb_nz^n\right]-(\beta-\alpha)\left[z+\sum_{n=2}^{\infty}a_nc_nz^n\right]$$

Comparing the coefficient of z^n on both sides of the above equation we get

$$a_n=\frac{1-\beta}{(1-\alpha)(b_n-c_n)}\left[p_{n-1}+a_2c_2p_{n-2}+a_3c_3p_{n-3}+......+a_{n-1}c_{n-1}p_1\right].$$

Taking modulus on both sides and applying $|p_n|\le 2 \quad \forall n$ we get

$$|a_n|\le\frac{2(1-\beta)}{(1-\alpha)(b_n-c_n)}\left[1+c_2|a_2|+c_3|a_3|+.......+c_{n-2}|a_{n-2}|+c_{n-1}|a_{n-1}|\right].$$

For n = 2 $|a_2|\le\dfrac{2(1-\beta)}{(1-\alpha)(b_2-c_2)}$ which proves the result (2.2).

For n = 3 $|a_3|\le\dfrac{2(1-\beta)}{(1-\alpha)(b_3-c_3)}\left[1+\dfrac{2(1-\beta)}{(1-\alpha)(b_2-c_2)}\cdot c_2\right]$

Thus, the result in (2.3) is true for n = 3. Suppose that the result in (2.3) is true for n = 4, 5, …..k.

For $n = k + 1$

Consider

$$|a_{k+1}|\le\frac{2(1-\beta)}{(1-\alpha)(b_{k+1}-c_{k+1})}\left[1+c_2\cdot\frac{2(1-\beta)}{(1-\alpha)(b_2-c_2)}+c_3\cdot\frac{2(1-\beta)}{(1-\alpha)(b_3-c_3)}\left[1+\frac{2(1-\beta)}{(1-\alpha)(b_2-c_2)}\cdot c_2\right]\right]$$

$$+........+\frac{2(1-\beta)}{(1-\alpha)(b_k-c_k)}\mathop{\pi}_{j=1}^{k-1}\left(1+\frac{2(1-\beta)}{(1-\alpha)(b_j-c_j)}\cdot c_j\right)$$

$$|a_{k+1}|\le\frac{2(1-\beta)}{(1-\alpha)(b_{j+1}-c_{j+1})}\mathop{\pi}_{j=1}^{k-1}\left(1+\frac{2(1-\beta)}{(1-\alpha)(b_{j+1}-c_{j+1})}.c_{j+1}\right)$$

Hence the result is true for $n = k + 1$. By mathematical induction the result (2.3) is true for all $n \geq 3$.

3. FEKETE – SZEGO INEQUALITY

Theorem: If $f(z) \in Q(\phi, \psi, \alpha, \beta)$ with $0 \leq \alpha \leq \beta < 1$ then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\beta)}{(1-\alpha)(b_3 - c_3)} \max\{1, |2v-1|\}. \tag{3.1}$$

Where $v = \dfrac{1-\beta}{(1-\alpha)(b_2 - c_2)^2}\left[\mu(b_3 - c_3) - c_2(b_2 - c_2)\right]$

And the result is sharp.

Proof: Since $f(z) \in Q(\phi, \psi, \alpha, \beta)$, from theorem (2.2) we have

$$a_2 = \frac{1-\beta}{(1-\alpha)(b_2 - c_2)}.P_1 \quad \text{and}$$

$$a_3 = \frac{1-\beta}{(1-\alpha)(b_3 - c_3)}\left[p_2 + a_2 c_2 p_1\right].$$

For any complex number μ we have

$$a_3 - \mu a_2^2 = \frac{1-\beta}{(1-\alpha)(b_3 - c_3)}\left[p_2 - v p_1^2\right] \tag{3.2}$$

Here $v = \dfrac{1-\beta}{(1-\alpha)(b_2 - c_2)^2}\left[\mu(b_3 - c_3) - c_2(b_2 - c_2)\right]$

By taking modulus on both sides equation (3.2) and applying lemma (1.1) to the right hand side we get the result (3.1). The result is sharp, that is

$$\left|a_3 - \mu a_2^2\right| = \frac{2(1-\beta)}{(1-\alpha)(b_3 - c_3)} \quad \text{if } P(z) - \frac{1+z^2}{1-z^2}$$

$$= \frac{2(1-\beta)}{(1-\alpha)(b_3 - c_3)}\left[\frac{2(1-\beta)[\mu(b_3 - c_3) - c_2(b_2 - c_2)] - (1-\alpha)(b_2 - c_2)}{(1-\alpha)(b_2 - c_2)^2}\right]$$

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If $P(z) = \frac{1+z}{1-z}$

Theorem 3.2: If $f(z) \in Q(\phi, \psi, \alpha, \beta)$, with $0 \le \alpha \le \beta < 1$ then for any real number μ we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\beta)}{(1-\alpha)(b_3-c_3)}\left\{\frac{1-\beta}{(1-\alpha)(b_2-c_2)^2}\left[4c_2(b_2-c_2) - 4\mu(b_3-c_3)\right] + 2 \quad \text{if } \mu \le \sigma_1\right\}$$

$$\le \frac{2(1-\beta)}{(1-\alpha)(b_3-c_3)} \qquad \text{if} \quad \sigma_1 \le \mu \le \sigma_2$$

$$\le \frac{(1-\beta)}{(1-\alpha)(b_3-c_3)}\left\{\frac{1-\beta}{(1-\alpha)(b_2-c_2)^2}\left[4\mu(b_3-c_3) - 4c_2(b_2-c_2)\right] - 2\right\} \quad \text{if } \mu \ge \sigma_2$$

(3.3)

Furthermore if $\sigma_1 \le \mu \le \sigma_3$

$$\left|a_3 - \mu a_2^2\right| + \left[\mu(b_3-c_3) - c_2(b_2-c_2)\right]\frac{|a_2|^2}{(b_3-c_3)} \le \frac{2(1-\beta)}{(1-\alpha)(b_3-c_3)}$$

If $\sigma_3 \le \mu \le \sigma_2$ then

$$\left|a_3 - \mu a_2^2\right| + (1-\alpha)(b_2-c_2) - (1-\beta)\left[\mu(b_3-c_3) - c_2(b_2-c_2)\right]\frac{|a_2|^2}{(1-\beta)(b_3-c_3)} \le \frac{2(1-\beta)}{(1-\alpha)(b_3-c_3)}.$$

Where $\sigma_1 = \frac{c_2(b_2-c_2)}{b_3-c_3}$

$$\sigma_2 = \frac{(1-\alpha)(b_2-c_2)^2 + c_2(b_2-c_2)(1-\beta)}{(1-\beta)(b_3-c_3)}$$

$$\sigma_3 = \frac{2(1-\beta)(b_2-c_2)c_2 + (1-\alpha)(b_2-c_2)^2}{2(1-\beta)(b_3-c_3)}.$$

Proof: Since $f \in Q(\phi, \psi, \alpha, \beta)$, from equation (3.2) we have

$$a_3 - \mu a_2^2 = \frac{1-\beta}{(1-\alpha)(b_3-c_3)}\left[p_2 - v p_1^2\right]$$

Where $v = \frac{1-\beta}{(1-\alpha)(b_2-c_2)^2}[\mu(b_3-c_3)-c_2(b_2-c_2)]$

Taking modulus on both sides of the above equation and applying Lemma (1.2) on the right hand side, we get the following cases

i) If $\mu \le \sigma_1$ then

$$\mu \le \frac{c_2(b_2-c_2)}{b_3-c_3}$$

After simplification, we get

$v \le 0 \Rightarrow \left|p_2 - vp_1^2\right| \le -4v+2$

$$\left|p_2 - vp_1^2\right| \le \frac{1-\beta}{(1-\alpha)(b_2-c_2)^2}[4c_2(b_2-c_2)-4\alpha(b_3-c_3)]+2 \qquad (3.4)$$

ii) if $\sigma_1 \le \mu \le \sigma_2$ then

$$\frac{c_2(b_2-c_2)}{b_3-c_3} \le \mu \le \frac{(1-\alpha)(b_2-c_2)^2 + c_2(b_2-c_2)(1-\beta)}{(1-\beta)(b_3-c_3)}.$$

Which on simplification, we get

$0 \le v \le 1$ implies

$$\left|p_2 - vp_1^2\right| \le 2 \qquad (3.5)$$

iii) If $\mu \ge \sigma_2$ then

$$\mu \ge \frac{(1-\alpha)(b_2-c_2)^2 + c_2(b_2-c_2)(1-\beta)}{(1-\beta)(b_3-c_3)}.$$

After simplification, we get

$v \ge 1$ implies

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$$\left|p_2 - vp_1^2\right| \le \frac{1-\beta}{(1-\alpha)(b_2-c_2)^2}\left[4\mu(b_3-c_3) - 4c_2(b_2-c_2)\right] - 2. \qquad (3.6)$$

From equations (3.2), (3.4) (3.5) and (3.6) we get the result in (3.3)

Case IV: If $\sigma_1 \le \mu \le \sigma_3$, then

$$\frac{c_2(b_2-c_2)}{b_3-c_3} \le \mu \le \frac{(1-\alpha)(b_2-c_2)^2 + 2(1-\beta)(b_2-c_2)c_2}{2(1-\beta)(b_3-c_3)}.$$

After simplification we get

$0 \le v \le \frac{1}{2}$ Implies

$$\left|p_2 - vp_1^2\right| + v\left|p_1\right|^2 \le 2$$

$$\frac{(1-\alpha)(b_3-c_3)}{1-\beta}\left|a_3 - \mu a_2^{\,2}\right| + \frac{(1-\beta)}{(1-\alpha)(b_2-c_2)^2}$$

$$\left[\mu(b_3-c_3) - c_2(b_2-c_2)\right]\cdot\frac{(1-\alpha)^2(b_2-c_2)^2}{(1-\beta)^2}\cdot\left|a_2\right|^2 \le 2.$$

This implies

$$\left|a_3 - \mu a_2^{\,2}\right| + \left[\mu(b_3-c_3) - c_2(b_2-c_2)\right]\frac{\left|a_2\right|^2}{(b_3-c_3)} \le \frac{2(1-\beta)}{(1-\alpha)(b_3-c_3)}.$$

Case V: If $\sigma_3 \le \mu \le \sigma_2$, then

$$\frac{(1-\alpha)(b_2-c_2)^2 + 2(1-\beta)c_2(b_2-c_2)}{2(1-\beta)(b_3-c_3)} \le \mu \le \frac{(1-\alpha)(b_2-c_2)^2 + (1-\beta)(b_2-c_2)c_2}{(1-\beta)(b_3-c_3)}$$

After simplification we get

$$\frac{1}{2} \le v \le 1$$

$$\Rightarrow \quad \left|p_2 - v p_1{}^2\right| + (1-v)\left|p_1\right|^2 \le 2$$

$$\left|a_3 - \mu a_2{}^2\right| + \frac{1}{(1-\beta)(b_3 - c_3)}\left[(1-\alpha)(b_2-c_2)^2 - (1-\beta)\left[\mu(b_3-c_3) - c_2(b_2-c_2)\right]\right]\left|a_2{}^2\right|$$

$$\le \frac{2(1-\beta)}{(1-\alpha)(b_3-c_3)}.$$

This completes proof the theorem.

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