FEKETE - SZEGO INEQUALITY FOR SOME SUB-CLASSES OF ANALYTIC FUNCTIONS INVOLVING COMPLEX ORDER

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Abstract: The aim of the present paper is to introduce a new sub-class of analytic functions involving complex order and obtain the Fekete – Szego inequality for the functions in this class. The Fekete – Szego inequality for the inverse function of f in this class is also obtained. Certain applications of our results for the functions defined through convolution are also obtained.

1. INTRODUCTION

Let A denote the class of all analytic functions *f* of the form,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad \left(z \in \Delta = \{z \in C / |z| < 1\}\right). \tag{1}$$

Let S be the sub-class of A consisting of univalent functions in Δ.

A function $f \in \mathcal{A}$ is subordinate to an univalent function $g \in \mathcal{A}$ written as $f \prec g$ if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

Let B_0 be the family of analytic functions $w(z)$ in the unit disk Δ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1 \quad \forall z \in \Delta$.

If $f \prec g$ then there is a function $w(z)$ in B_0 such that $f(z) = g(w(z))$.

Definition (1): Let $\phi(z)$ be a univalent analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis.

Such a function ϕ has a series expansion of the form, $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ with $B_1 > 0,\ B_2 \geq 0$ and B_n's are real.

Let $\mathbb{S}^*(\phi)$ be the class of functions $f \in \mathbb{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \qquad (z \in \Delta)$$

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and $\mathbb{C}(\phi)$ be the class of functions $f \in \mathbb{S}$ for which

$$1+\frac{zf''(z)}{f'(z)} \prec \phi(z), \qquad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions.
These classes were introduced and studied by W.Ma and D.Minda [1]. They have obtained the Fekete - Szego inequality for functions in the classes $\mathbb{S}^*(\phi)$ and $\mathbb{C}(\phi)$.

V. Ravichandran, M. Bolcal, Y.Polotaglu and A.Sen [2] have further generalised these classes by defining $\mathbb{S}_b^*(\phi)$ to be the class of functions $f \in \mathbb{S}$ for which,

$$1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right] \prec \phi(z), \qquad (z \in \Delta)$$

and

$\mathbb{C}_b(\phi)$ be the class of functions $f \in \mathbb{S}$ for which,

$$1+\frac{1}{b}\left[\frac{zf''(z)}{f'(z)}\right] \prec \phi(z), \qquad (z \in \Delta)$$

Where *b* is a non-zero complex number.

Recently T.N. Shanmugam and S. Sivasubramanian [3] have defined the class $M_\alpha(\phi)$ to be the class of all functions $f \in \mathcal{A}$ which satisfies the condition,

$$\frac{\alpha z^2 f''(z)+zf'(z)}{(1-\alpha)f(z)+\alpha zf'(z)} \prec \phi(z) \quad \forall z \in \Delta.$$

Very recently T.N. Shanmugam, S. Sivasubramanian and B.A.Frasin [4] have defined and studied the class $M[b,\alpha](\phi)$ consisting of functions $f \in \mathcal{A}$ satisfying the condition,

$$1+\frac{1}{b}\left(\frac{\alpha z^3 f'''(z)+(1+2\alpha)z^2 f''(z)+zf'(z)}{\alpha z^2 f''(z)+zf'(z)}-1\right) \prec \phi(z), \quad z \in \Delta,$$

where b is non-zero complex number and $0 \le \alpha < 1$.

All these authors have obtained the Fekete – Szego inequality for the functions in these classes. They have also found certain applications of their results for the functions defined through convolution and fractional derivatives.

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In this present paper, we define a sub-class of analytic function involving complex order and obtain the Fekete - Szego inequality for the functions in this class. Our results unify and generalise the recent results of W.Ma and D.Minda [1], V.Ravichandran, Y.Polatoglu, M.Bolcal and A.Sen [2], T.N. Shanmugam and S.Sivasubramanian [3], T.N.Shanmugam, S.Sivasubramanian and B.A.Frasin [4].

Definition (2):- Let 'b' be a non-zero complex number and γ, α be real numbers with $0 < \gamma \leq 1$, $0 \leq \alpha < 1$ and $\phi(z)$ is a function as in the definition (1). Then a function $f \in \mathcal{A}$ is in the class $M[b,\alpha,\gamma](\phi)$ if,

$$1+\frac{1}{b}\left(\frac{\alpha z^3 f'''(z)+(1+2\alpha)z^2 f''(z)+zf'(z)}{\alpha z^2 f''(z)+zf'(z)}-1\right) \prec [\phi(z)]^{\gamma}, \quad \forall z \in \Delta.$$

Where the powers are taken with their principle values. It is noted that,

1) $M[b,\alpha,1](\phi) = M[b,\alpha]$ defined and studied by T.N. Shanmugam, S.Sivasubramanian and B.A.Frasin [4].
2) $M[b,0,1](\phi) = C_b[\phi]$ defined and studied by V. Ravichandran, M.Bolcol, Y. Polatoglu and A. Sen [2].
3) $M[1,0,1](\phi) = C[\phi]$ defined and studied by Ma and Minda [1].

Definition (3): If b, γ, α and ϕ are same as in the definition (2) then for any fixed function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n \in \mathcal{A}$ $(g_n > 0)$, let $M[b,\alpha,\gamma,g](\phi)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in M[b,\alpha,\gamma](\phi)$.

It is noted that $M\left[b,\alpha,\gamma,\frac{z}{1-z}\right](\phi) = M[b,\alpha,\gamma](\phi)$.

Definition (4): Let b, γ, α and ϕ be same as in the definition (2) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M[b,\alpha,\gamma](\phi)$ is an invertible function and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ with $|w| < r_0$ is the inverse function of *f*, then

$$f^{-1}(f(z)) = z = f(f^{-1}(z)).$$

where r_0 is greater than the radius of the Koebe domain for the class $M[b,\alpha,\gamma](\phi)$.

In order to prove our main result, we require the following two Lemmas.

Lemma (1): [2]: If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is a function with positive real part, then for any complex number □,

$\left|c_2 - \mu c_1^2\right| \le 2 \max\left\{1, \left|2\mu - 1\right|\right\}$ and the result is sharp for the functions defined by $p(z) = \dfrac{1+z^2}{1-z^2}$ or $p(z) = \dfrac{1+z}{1-z}$.

Lemma (2): [1]: If $p_1(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$ is an analytic function with positive real part in □, then for any real number v, we have,

$$\left|c_2 - v c_1^2\right| \le -4v + 2 \quad \text{if} \quad v \le 0$$
$$\le 2 \quad \text{if} \quad 0 \le v \le 1$$
$$\le 4v - 2 \quad \text{if} \quad v \ge 1$$

when, $v < 0$ or $v > 1$, the equality holds if and only if $P_1(z)$ is $\dfrac{1+z}{1-z}$ or one of its rotations.

If $0<v<1$ then the equality holds if and only if $P_1(z)$ is $\dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v=0$ then the equality holds if and only if,

$P_1(z) = \left(\dfrac{1+\lambda}{2}\right)\left(\dfrac{1+z}{1-z}\right) + \left(\dfrac{1-\lambda}{2}\right)\left(\dfrac{1-z}{1+z}\right) \qquad (0 \le \lambda \le 1)$ or one of its rotations.

If $v=1$ the equality holds only for the reciprocal of $P_1(z)$ for the case $v=0$.
Also the above upper bound is sharp and it can be further improved as follows, when $0<v<1$,

$$\left|c_2 - v c_1^2\right| + v\left|c_1^2\right| \le 2 \qquad \left(0 < v \le \frac{1}{2}\right)$$

$$\left|c_2 - v c_1^2\right| + (1-v)\left|c_1^2\right| \le 2 \qquad \left(\frac{1}{2} < v \le 1\right)$$

2. FEKETE – SZEGO INEQUALITY:

In this section we prove the Fekete – Szego inequality for the functions in the classes $M[b,\alpha,\gamma](\phi)$, $M[b,\alpha,\gamma,g](\phi)$ and for the inverse function of *f* in the class $M[b,\alpha,\gamma](\phi)$.

Theorem (1):- If $f \in M[b,\alpha,\gamma](\phi)$ then for any complex number '□', we have,

$$\left|a_3-\mu a_2{}^2\right| \le \frac{|b|\gamma B_1}{6(1+2\alpha)}\max\left\{1,\left|\frac{B_2}{B_1}+\left(\frac{\gamma-1}{2}\right)B_1-b\gamma B_1\delta\right|\right\}, \tag{2}$$

where $\delta=\frac{3}{2}\frac{(1+2\alpha)}{(1+\alpha)^2}\mu-1$ and the result is sharp.

Proof:- Since $f \in M[b,\alpha,\gamma](\phi)$ then by the definition (2), we have,

$$1+\frac{1}{b}\left(\frac{\alpha z^3 f'''(z)+(1+2\alpha)z^2 f''(z)+zf'(z)}{\alpha z^2 f''(z)+zf'(z)}-1\right)\prec[\phi(z)]^\gamma,\quad \forall z\in\Delta.$$

Then by subordination principle, there exists a Schwartz's function $w(z)$ in □ with $w(0)=0$ and $|w(z)|<1$, such that,

$$1+\frac{1}{b}\left(\frac{\alpha z^3 f'''(z)+(1+2\alpha)z^2 f''(z)+zf'(z)}{\alpha z^2 f''(z)+zf'(z)}-1\right)=[\phi(w(z))]^\gamma,\quad \forall z\in\Delta. \tag{3}$$

Now define a function $P(z)$, such that,

$$p(z)=1+\frac{1}{b}\left(\frac{\alpha z^3 f'''(z)+(1+2\alpha)z^2 f''(z)+zf'(z)}{\alpha z^2 f''(z)+zf'(z)}-1\right)=1+b_1z+b_2z^2+\dots,\quad \forall z\in\Delta$$

$$\alpha z^3 f'''(z)+(1+2\alpha)z^2 f''(z)+zf'(z)=\left[1+(bb_1)z+(bb_2)z^2+\dots\right]\left[\alpha z^2 f''(z)+zf'(z)\right] \quad \dots(4)$$

Replacing $f'(z), f''(z)$ and $f'''(z)$ with their equivalent expressions in series on both sides of (4) and comparing the coefficients of z^2 and z^3, we get,

$$a_2=\frac{bb_1}{2(1+\alpha)} \tag{5}$$

$$a_3=\frac{b}{6(1+2\alpha)}\left[b_2+bb_1{}^2\right] \tag{6}$$

Define another function $P_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\dots$ where $P_1(z)$ is analytic in □ with $P_1(0)=1$ and $\{Re\,P_1(z)\}>0$.

Solving $w(z)$ in terms of $P_1(z)$, we get,

$$w(z)=\frac{1}{2}\left(c_1 z+\left(c_2-\frac{c_1^{\,2}}{2}\right)z^2+\ldots\right).$$

$$\phi(w(z))=1+\left(\frac{B_1 c_1}{2}\right)z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^{\,2}}{2}\right)+\frac{1}{4}B_2 c_1^{\,2}\right]z^2+\ldots$$

$$\left[\phi(w(z))\right]^{\gamma}=1+\left(\frac{\gamma B_1 c_1}{2}\right)z+\left[\frac{\gamma B_1}{2}\left(c_2-\frac{c_1^{\,2}}{2}\right)+\frac{\gamma}{4}B_2 c_1^{\,2}+\frac{\gamma(\gamma-1)}{2}B_1^{\,2}\frac{c_1^{\,2}}{4}\right]z^2+\ldots$$

As $P(z)=\left[\phi(w(z))\right]^{\gamma}$ we get,

$$1+b_1 z+b_2 z^2+b_3 z^3+\ldots$$

$$=1+\left(\frac{\gamma B_1 c_1}{2}\right)z+\left[\frac{\gamma B_1}{2}\left(c_2-\frac{c_1^{\,2}}{2}\right)+\frac{\gamma}{4}B_2 c_1^{\,2}+\frac{\gamma(\gamma-1)}{2}B_1^{\,2}\frac{c_1^{\,2}}{4}\right]z^2+\ldots \tag{7}$$

Now comparing the coefficients of z and z^2 on both sides of the above equation (7), we get,

$$b_1=\frac{\gamma B_1 c_1}{2} \tag{8}$$

and

$$b_2=\frac{\gamma B_1 c_2}{2}-\frac{\gamma B_1 c_1^{\,2}}{4}+\frac{\gamma B_2 c_1^{\,2}}{4}+\frac{\gamma(\gamma-1)}{2}\,\frac{B_1^{\,2}c_1^{\,2}}{4}. \tag{9}$$

From the equations (5), (6), (8) and (9), we get,

$$a_2=\frac{b\gamma B_1 c_1}{4(1+\alpha)} \tag{10}$$

and

$$a_3=\frac{b\gamma B_1}{12(1+2\alpha)}\left[c_2-\frac{c_1^{\,2}}{2}\left\{1-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1-b\gamma B_1\right\}\right] \tag{11}$$

For any complex number '□', we get,

$$a_3-\mu a_2^{\,2}=\frac{b\gamma B_1}{12(1+2\alpha)}\left[c_2-v c_1^{\,2}\right] \tag{12}$$

where, $v=\frac{1}{2}\left[1-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1+b\gamma B_1\delta\right]$

and $\delta = \frac{3}{2}\frac{(1+2\alpha)}{(1+\alpha)^2}\mu - 1$

Now by taking modulus to the equation (12) and applying Lemma (1) to the R.H.S. of (12), we get,

$$\left|a_3 - \mu a_2{}^2\right| \le \frac{|b|\,\gamma B_1}{6(1+2\alpha)} \max\left\{1, \left|\frac{B_2}{B_1} + \left(\frac{\gamma-1}{2}\right)B_1 - b\gamma B_1\delta\right|\right\}, \tag{13}$$

This completes the proof of the theorem.
And the result is sharp, for the functions i.e.

$$P(z) = \left(\frac{1+z^{\ 2}}{1-z^2}\right)^{\gamma} \text{ and } \left(\frac{1+z}{1-z}\right)^{\gamma}$$

Note: By taking b =1 and considering □ as real number and by applying Lemma (1.2), we now obtain Fekete – Szego inequality for the functions in the class $M[\alpha,\gamma](\phi)$.

Theorem (2):- If $f \in M[\alpha,\gamma](\phi)$ then for any real number '□' and for,

$$\sigma_1 = \frac{2}{3}\left[\frac{(B_2 - B_1) + \left(\frac{\gamma-1}{2}\right)B_1{}^2 + \gamma B_1{}^2}{\gamma B_1{}^2}\right]\frac{(1+\alpha)^2}{(1+2\alpha)}$$

$$\sigma_2 = \frac{2}{3}\left[\frac{(B_2 + B_1) + \left(\frac{\gamma-1}{2}\right)B_1{}^2 + \gamma B_1{}^2}{\gamma B_1{}^2}\right]\frac{(1+\alpha)^2}{(1+2\alpha)}$$

$$\sigma_3 = \frac{2}{3}\left[\frac{B_2 + \left(\frac{\gamma-1}{2}\right)B_1{}^2 + \gamma B_1{}^2}{\gamma B_1{}^2}\right]\frac{(1+\alpha)^2}{(1+2\alpha)}$$

$\delta = \frac{3}{2}\frac{(1+2\alpha)}{(1+\alpha)^2}\mu - 1$ then,

$$\left|a_3-\mu a_2^2\right|\leq\frac{\gamma B_1}{6(1+2\alpha)}\left[\frac{B_2}{B_1}+\left(\frac{\gamma-1}{2}\right)B_1-\gamma B_1\delta\right] \text{ if } \mu\leq\sigma_1$$

$$\leq\frac{\gamma B_1}{6(1+2\alpha)} \qquad \text{if} \qquad \sigma_1\leq\mu\leq\sigma_2$$

$$\leq\frac{\gamma B_1}{6(1+2\alpha)}\left[-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1+\gamma B_1\delta\right] \text{ if } \mu\geq\sigma_2$$

Further more if $\sigma_1\leq\mu\leq\sigma_3$ then

$$\left|a_3-\mu a_2^2\right|+\frac{2}{3}\frac{(1+\alpha)^2}{(1+2\alpha)}\left[1-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1+\gamma B_1\delta\right]\frac{|a_2|^2}{\gamma B_1}\leq\frac{\gamma B_1}{6(1+2\alpha)}$$

and if $\sigma_3\leq\mu\leq\sigma_2$ then

$$\left|a_3-\mu a_2^2\right|+\frac{2}{3}\frac{(1+\alpha)^2}{(1+2\alpha)}\left[-1+\frac{B_2}{B_1}+\left(\frac{\gamma-1}{2}\right)B_1-\gamma B_1\delta\right]\frac{|a_2|^2}{\gamma B_1}\leq\frac{\gamma B_1}{6(1+2\alpha)} \tag{14}$$

Proof:- If $f\in M[\alpha,\gamma](\phi)$ then by proceeding as in theorem (1) with b=1, and for any real number □, we get,

$$a_3-\mu a_2^2=\frac{\gamma B_1}{12(1+2\alpha)}\left[c_2-vc_1^2\right] \tag{15}$$

where

$$v=\frac{1}{2}\left[1-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1+\gamma B_1\delta\right]$$

and $\delta=\frac{3}{2}\frac{(1+2\alpha)}{(1+\alpha)^2}\mu-1.$

Taking modulus on both sides of equation (15) and by applying Lemma(2) to the R.H.S. of (15), we have the following cases:

Case 1. If $\mu\leq\sigma_1$ then

$$\mu\leq\frac{\frac{2}{3}\left[(B_2-B_1)+\left(\frac{\gamma-1}{2}\right)B_1^2+\gamma B_1^2\right]}{\gamma B_1^2}\frac{(1+\alpha)^2}{(1+2\alpha)}$$

which on simplification, we get,

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$$\frac{1}{2}\left[1-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1+\gamma B_1\delta\right]\le 0$$

$$\Rightarrow v\le 0$$

$$\Rightarrow \left|c_2-vc_1{}^2\right|\le -4v+2$$

$$\Rightarrow \left|c_2-vc_1{}^2\right|\le 2\frac{B_2}{B_1}+(\gamma-1)B_1-2\gamma B_1\delta \tag{16}$$

where $\delta=\frac{3}{2}\frac{(1+2\alpha)}{(1+\alpha)^2}\mu-1$

Case 2. If $\sigma_1\le\mu\le\sigma_2$ then

$$\frac{\frac{2}{3}\left[(B_2-B_1)+\left(\frac{\gamma-1}{2}\right)B_1{}^2+\gamma B_1{}^2\right]}{\gamma B_1{}^2}\frac{(1+\alpha)^2}{(1+2\alpha)}\le\mu$$

$$\le\frac{\frac{2}{3}\left[(B_2+B_1)+\left(\frac{\gamma-1}{2}\right)B_1{}^2+\gamma B_1{}^2\right]}{\gamma B_1{}^2}\frac{(1+\alpha)^2}{(1+2\alpha)}$$

which on simplification, we get,

$$0\le\frac{1}{2}\left[1-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1+\gamma B_1\delta\right]\le 1$$

$$\Rightarrow 0\le v\le 1$$

$$\Rightarrow \left|c_2-vc_1{}^2\right|\le 2 \tag{17}$$

Case 3. If $\mu\ge\sigma_2$ then

$$\mu\ge\frac{\frac{2}{3}\left[(B_2+B_1)+\left(\frac{\gamma-1}{2}\right)B_1{}^2+\gamma B_1{}^2\right]}{\gamma B_1{}^2}\frac{(1+\alpha)^2}{(1+2\alpha)}$$

which on simplification, we get,

$$\frac{1}{2}\left[1-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1+\gamma B_1\delta\right]\ge 1$$

$$\Rightarrow v\ge 1$$

$$\Rightarrow \left|c_2 - vc_1^{\,2}\right| \le 4v - 2$$

$$\Rightarrow \left|c_2 - vc_1^{\,2}\right| \le -2\frac{B_2}{B_1} - (\gamma - 1)B_1 + 2\gamma B_1\delta \tag{18}$$

From equations (15), (16), (17) and (18), we get,

$$\left|a_3 - \mu a_2^{\,2}\right| \le \frac{\gamma B_1}{6(1+2\alpha)}\left[\frac{B_2}{B_1} + \left(\frac{\gamma-1}{2}\right)B_1 - \gamma B_1\delta\right] \text{ if } \quad \mu \le \sigma_1$$

$$\le \frac{\gamma B_1}{6(1+2\alpha)} \quad \text{ if } \quad \sigma_1 \le \mu \le \sigma_2$$

$$\le \frac{\gamma B_1}{6(1+2\alpha)}\left[-\frac{B_2}{B_1} - \left(\frac{\gamma-1}{2}\right)B_1 + \gamma B_1\delta\right] \text{ if } \quad \mu \ge \sigma_2$$

Case 4. Further more if $\sigma_1 \le \mu \le \sigma_3$ then

$$\frac{\frac{2}{3}\left[(B_2 - B_1) + \left(\frac{\gamma-1}{2}\right)B_1^{\,2} + \gamma B_1^{\,2}\right]}{\gamma B_1^{\,2}} \frac{(1+\alpha)^2}{(1+2\alpha)} \le \mu$$

$$\le \frac{\frac{2}{3}\left[B_2 + \left(\frac{\gamma-1}{2}\right)B_1^{\,2} + \gamma B_1^{\,2}\right]}{\gamma B_1^{\,2}} \frac{(1+\alpha)^2}{(1+2\alpha)}$$

which on simplification, we get,

$$0 \le \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \left(\frac{\gamma-1}{2}\right)B_1 + \gamma B_1\delta\right] \le \frac{1}{2}$$

$$\Rightarrow 0 \le v \le \frac{1}{2}$$

$$\Rightarrow \left|c_2 - vc_1^{\,2}\right| + v\left|c_1^{\,2}\right| \le 2$$

$$\left|a_3 - \mu a_2^{\,2}\right| + \frac{2}{3}\frac{(1+\alpha)^2}{(1+2\alpha)}\left[1 - \frac{B_2}{B_1} - \left(\frac{\gamma-1}{2}\right)B_1 + \gamma B_1\delta\right]\frac{\left|a_2\right|^2}{\gamma B_1} \le \frac{\gamma B_1}{6(1+2\alpha)}$$

Case 5. If $\sigma_3 \le \mu \le \sigma_2$ then,

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$$\frac{\frac{2}{3}\left[B_2+\left(\frac{\gamma-1}{2}\right)B_1^{\,2}+\gamma B_1^{\,2}\right]}{\gamma B_1^{\,2}}\frac{(1+\alpha)^2}{(1+2\alpha)}\le\mu$$

$$\le\frac{\frac{2}{3}\left[(B_2+B_1)+\left(\frac{\gamma-1}{2}\right)B_1^{\,2}+\gamma B_1^{\,2}\right]}{\gamma B_1^{\,2}}\frac{(1+\alpha)^2}{(1+2\alpha)}$$

which on simplification, we get,

$$\frac{1}{2}\le\frac{1}{2}\left[1-\frac{B_2}{B_1}-\left(\frac{\gamma-1}{2}\right)B_1+\gamma B_1\delta\right]\le 1$$

$$\Rightarrow\frac{1}{2}\le v\le 1$$

$$\Rightarrow\left|c_2-vc_1^{\,2}\right|+(1-v)\left|c_1^{\,2}\right|\le 2$$

$$\left|a_3-\mu a_2^{\,2}\right|+\frac{2}{3}\frac{(1+\alpha)^2}{(1+2\alpha)}\left[1+\frac{B_2}{B_1}+\left(\frac{\gamma-1}{2}\right)B_1-\gamma B_1\delta\right]\frac{|a_2|^2}{\gamma B_1}\le\frac{\gamma B_1}{6(1+2\alpha)}$$

This completes the proof of the theorem.

Theorem (3):- If $f(z)=z+\sum_{n=2}^{\infty}a_n z^n\in M[b,\alpha,\gamma,g](\phi)$ then for any complex number □, we have,

$$\left|a_3-\mu a_2^{\,2}\right|\le\frac{|b|\gamma B_1}{6(1+2\alpha)g_3}\max\left\{1,\left|\frac{B_2}{B_1}+\left(\frac{\gamma-1}{2}\right)B_1-b\gamma B_1\delta_1\right|\right\},$$

…(19)

where $\delta_1=\frac{3}{2}\frac{(1+2\alpha)}{(1+\alpha)^2}\mu\frac{g_3}{g_2^{\,2}}-1$ and the result is sharp,

Proof: The proof of this theorem is similar to that of Theorem (1) and hence the details are omitted here.

Theorem (4):- If $f(z)=z+\sum_{n=2}^{\infty}a_n z^n\in M[b,\alpha,\gamma](\phi)$ and $f^{-1}(w)=w+\sum_{n=2}^{\infty}d_n w^n$ is the inverse function of *f* with $|w|<r_0$ where r_0 is

greater than the radius of the Koebe domain for the class $M[b,\alpha,\gamma](\phi)$ then for any complex number '□', we have,

$$\left|d_3-\mu d_2^2\right|\leq\frac{|b|\gamma B_1}{6(1+2\alpha)}\max\left\{1,\left|\frac{B_2}{B_1}+\left(\frac{\gamma-1}{2}\right)B_1-b\gamma B_1\delta_2\right|\right\}, \quad \ldots(20)$$

where $\delta_2=\frac{3}{2}\frac{(1+2\alpha)}{(1+\alpha)^2}(2-\mu)-1$ and the result is sharp.

Proof:- Since $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in M[b,\alpha,\gamma](\phi)$, then from the theorem (1), we get,

$$\left|a_3-\mu a_2^2\right|\leq\frac{|b|\gamma B_1}{6(1+2\alpha)}\max\left\{1,\left|\frac{B_2}{B_1}+\left(\frac{\gamma-1}{2}\right)B_1-b\gamma B_1\delta\right|\right\}, \quad (21)$$

where $\delta=\frac{3}{2}\frac{(1+2\alpha)}{(1+\alpha)^2}\mu-1$ and the result is sharp.

As $f^{-1}(w)=w+\sum_{n=2}^{\infty}d_nw^n$ is the inverse function of *f*, then by the definition of inverse function, we get,

$$f^{-1}(f(z))=z=f\left(f^{-1}(z)\right),$$

consider $f^{-1}(f(z))=z$.

$$\Rightarrow f^{-1}\left(z+\sum_{n=2}^{\infty}a_nz^n\right)=z.$$

$$\Rightarrow\left(z+a_2z^2+a_3z^3+\ldots\right)+\sum_{n=2}^{\infty}d_n\left(z+a_2z^2+a_3z^3+\ldots\right)^n=z,$$

which gives that,

$$z+(a_2+d_2)z^2+(a_3+2a_2d_2+d_3)z^3+\ldots=z. \quad 22)$$

Comparing the coefficients of z^2 and z^3 on both sides of equation (22), we get,

$$d_2=-a_2. \quad (23)$$

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$$d_3 = -a_3 + 2a_2^{\ 2}. \tag{24}$$

For any complex number '□', we get,

$$\left|d_3 - \mu d_2^{\ 2}\right| = \left|a_3 - (2-\mu)\, a_2^{\ 2}\right|,$$

$$\left|d_3 - \mu d_2^{\ 2}\right| = \left|a_3 - t a_2^{\ 2}\right|, \tag{25}$$

where $t = 2-\mu$, t is a complex number.

From equations (2) and (25), we get,

$$\left|d_3 - \mu d_2^{\ 2}\right| \le \frac{|b|\,\gamma B_1}{6(1+2\alpha)} \max\left\{1, \left|\frac{B_2}{B_1} + \left(\frac{\gamma-1}{2}\right) B_1 - b\gamma B_1 \delta_2\right|\right\},$$

where $\delta_2 = \dfrac{3}{2}\dfrac{(1+2\alpha)}{(1+\alpha)^2}(2-\mu) - 1$.

This completes the proof of the theorem.
And the result is sharp for the functions defined by

$$P(z) = \left(\frac{1+z^2}{1-z^2}\right)^{\gamma} \text{ and } \left(\frac{1+z}{1-z}\right)^{\gamma}.$$

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