MULTIVALUED FIXED POINT THE OREMS FOR MEIRKEELER TYPE CONTRACTIONS IN PARTIALLY ORDERED METRIC SPACE

Rajesh Kumar Saini[1]

Abstract: In this paper we introduce Meir-Keeler type contractions for multivalued mappings and prove some fixed point theorems in partially ordered metric spaces. Fixed point theory in partially ordered metric spaces has greatly developed in recent times. Our results extend, complement and unify some recent results.

Keywords: Fixed point theorem, Multivalued mapping; Meir-Keelar type contraction, Partial ordering.

1. INTRODUCTION

A number of authors have defined contractive type mappings on a complete metric space X which are generalization of well known Banach contraction:

Definition 1.1: Let (X, d) be a metric space and T: X → X a self mapping. If (X, d) is complete and T is a contraction, i.e., there exists a constant a ∈ [0, 1) such that

$$d(Tx, Ty) \leq ad(x, y), \text{ for all } x, y \in X, \tag{1.1}$$

then, by Banach contraction mapping principle, which is a classical and powerful tool in nonlinear analysis, we know that T has a unique fixed point p and, for any $x_0 \in X$, the Picard iteration $\{T^n x_0\}$ converges to p and which has the property that each such mapping has a unique fixed point. In 1969, Meir and Keeler [16] obtained a remarkable generalization of Banach contraction principle as follows:

Definition 1.2: A self map T: X → X of a metric space (X, d) is called on (ε-δ) contraction, for all ε > 0 there exists δ(ε) > 0 such that

$$x, y \in X, \varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow (Tx, Ty) < \varepsilon. \tag{1.2}$$

Recently, Ran and Reurings [23] have initiated another important direction in generalizing the Banach contraction mapping principle in partial ordering on the metric space (X, d) and by requiring that the contraction condition (1.1) is satisfied only for comparable elements, that is, we have

$$d(Tx, Ty) \leq ad(x, y), \text{for all } x, y \in X, \text{ with } x \geq y. \tag{1.3}$$

Meir and Keeler [16] proved that an (ε-δ) contraction of a complete metric space X has a unique fixed point in X. There exists a vast literature which generalize the result of Meir and Keeler (see [9, 14, 17, 18, 19, 20, 25, 26] and references therein).

We will use the following relation between two nonempty subsets of a partially ordered set.

Definition 1.3 [25]: Let A and B be two nonempty subsets of a partially ordered set $(X; \leq)$. The relation between A and B is denoted and defined as follows:

$A \leq B$, if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

We will utilize the following control function which is also referred to as altering distance function.

Definition 1.4 [2]: A function $\psi: [0, \infty) \to [0, \infty)$ is called an Altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

For the use of control function in metric fixed point theory see some recent references ([2], [3],)

2. Preliminaries: Throughout this paper, let (X, d) be a metric space unless mentioned otherwise and B(X) is the set of all non-empty bounded subsets of X. Let $\delta(A, B)$ and $D(A, B)$ be the functions defined by

$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$

$D(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$

for all A, B in B(X). If A is a singleton i.e. $A = \{a\}$, we write

$\delta(A, B) = \delta(a, B)$ and $D(A, B) = D(a, B)$

If B is also a singleton i.e. $B = \{b\}$, we write

$\delta(A, B) = \delta(A, b)$ and $D(A, B) = D(A, b)$

It is obvious that $D(A, B) \leq \delta(A, B)$. For all $A, B, C \in B(X)$. The definition of $\delta(A, B)$ yields the following:

$\delta(A, B) = \delta(B, A) \geq 0$,

$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$,

$\delta(A, B) = 0$ iff $A = B = \{a\}$ and $\delta(A, A) = \text{diam } A$,

Definition 2.1 [7]: A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for n = 1,2,........, and $\{a_n\}$ converges to a.

given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centres in A and radius ε.

Lemma 2.1 [6]: If $\{A_n\}$ and $\{B_n\}$ are sequences in B(X) converging to A and B in B(X), respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.

Lemma 2.2 [7]: Let $\{A_n\}$ be a sequence in B(X) and y a point in X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in B(X).

In [11], Jungck and Rhoades extended definition of compatibility to set valued mappings setting as follows.

Definition 2.2: The mapping $I : X \to X$ and $f : X \to B(X)$ are $\tilde{\delta}$compatible if
$\lim_{n\to\infty} \delta(fIx_n, Ifx_n) = 0$,
whenever $\{x_n\}$ is a sequence in X such that If $x_n \in B(X)$, $fx_n \to \{t\}$ and $Ix_n \to t$,
for some $t \in X$.
Recently, the following definition is given by Jungck and Rhoades [13].

Definition 2.3: The mapping $I: X \to X$ and $f: X \to B(X)$ are weakly compatible if for each point u in X such that
$fu = \{Iu\}$, we have $fIu = Ifu$.

It can be seen that any δ-compatible mappings are weakly compatible but the converse is not true as shown by an example in [22].

3. MAIN RESULT: WE PROVE THE FOLLOWING THEOREM:

Theorem 3.1: Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $I, J: X \to X$ be single valued and $F, G: X \to CB(X)$ be multivalued mappings such that the following conditions are satisfied

(i) $\cup F(X) \sqsubseteq J(X)$ and $\cup G(X) \sqsubseteq I(X)$

(ii) {F, I} and {G, J} are weakly compatible,

(iii) either for given $\varepsilon > 0$ there exists $\varepsilon + \delta(\varepsilon) > 0$ such that

$\varepsilon \leq M(x, y) < \varepsilon + \delta(\varepsilon)$ implies $\psi(\delta(Fx, Gy)) < \varepsilon$

where $M(x, y) = \psi(\max \{d(Ix, Jy), D(Ix, Fx), D(Jy, Gy), \frac{D(Ix, Gy) + D(Iy, Fx)}{2}\})$, or $M(x, y) = 0$ implies diam Fx = diam Gy.

For all comparable $x, y \in X$ and ψ is an altering distance function and suppose that one of $I(X)$ or $J(X)$ is complete. Then there exists a unique point $p \in X$ such that

$Fp = Gp = \{Ip\} = \{Jp\} = \{p\}$

Proof: Let x_0 be an arbitrary point of X. By (i) we choose a point $x_1 \in X$ such that $y_1 = Fx_0 \sqsubseteq Jx_1$. For this point x_1, there exists a point $x_2 \in X$ such that $y_2 = Gx_1 \sqsubseteq Ix_2$, and so on. Continuing in this manner we can define a sequence $\{y_n\}$ as follows

$y_{2n+1} = Fx_{2n} \sqsubseteq Jx_{2n+1}$, $y_{2n+2} = Gx_{2n+1} \sqsubseteq Ix_{2n+2}$. (3.1)

We claim that $\{y_n\}$ is a Cauchy sequence. Two cases arise. Either $y_n = y_{n+1}$ for some n or $y_n \neq y_{n+1}$ for each n. If $y_n = y_{n+1}$ for some n then, $y_n = y_{n+k}$ for each $k \geq 1$. For instance suppose $y_{2m+1} = y_{2m+2}$. Then $y_{2m+2} = y_{2m+3}$. Otherwise using (iii), we get

$$\varepsilon \leq \psi(\max\{d(Ix_{2m+1}, Jx_{2m+2}), D(Ix_{2m+1}, Fx_{2m+1}),$$

$$D(Jx_{2m+2}, Gx_{2m+2}), \frac{D(Ix_{2m+1}, Gx_{2m+2}) + D(Ix_{2m+2}, Fx_{2m+1})}{2}\}),$$

$$\leq \psi(\max\{d(y_{2m+1}, y_{2m+2}), d(y_{2m+1}, y_{2m+2}), d(y_{2m+2}, y_{2m+3}),$$

$$\frac{d(y_{2m+1}, y_{2m+3}) + d(y_{2m+2}, y_{2m+2})}{2}\}),$$

Since

$$\frac{d(y_{2m+1}, y_{2m+3})}{2} \leq \max\{d(y_{2m+1}, y_{2m+2}), d(y_{2m+2}, y_{2m+3})\},$$

It follows that

$$\varepsilon \leq \psi(\max\{d(y_{2m+1}, y_{2m+2}), d(y_{2m+2}, y_{2m+3})\}) \tag{3.2}$$

Suppose that if $d(y_{2m+1}, y_{2m+2}) \geq d(y_{2m+2}, y_{2m+3})$, for some positive integer n, then from (3.2), we have

$$\varepsilon \leq \psi(d(y_{2m+1}, y_{2m+2})) < d(y_{2m+1}, y_{2m+2}) < \varepsilon + \delta(\varepsilon)$$

which implies that $\psi(d(y_{2m+1}, y_{2m+2})) < \varepsilon$

Hence $y_{2m+2} = y_{2m+3}$. Again if $d(y_{2m+1}, y_{2m+2}) \leq d(y_{2m+2}, y_{2m+3})$, for each positive integer n, then from (3.2), we have

$$\varepsilon \leq \psi(d(y_{2m+2}, y_{2m+3})) < d(y_{2m+2}, y_{2m+3}) < \varepsilon + \delta(\varepsilon)$$

implies $\psi(d(y_{2m+3}, y_{2m+4})) < d(y_{2m+3}, y_{2m+4}) < \varepsilon$,

i.e. $y_{2m+3} = y_{2m+4}$.

Proceeding in this manner, it follows that $y_{2m+1} = y_{2m+k}$ for each $k > 1$ and $\{y_n\}$ is a Cauchy sequence. Let us, therefore, consider the case when $y_n \neq y_{n+1}$ for each n. In this case, using (iii), we obtain

$$\varepsilon \leq M(x_n, x_{n+1}) \leq \varepsilon + \delta(\varepsilon), \text{ implies } d(y_{n+1}, y_{n+2}) < \varepsilon.$$

Thus $\{d(y_n, y_{n+1})\}$ is strictly decreasing sequence of positive numbers and therefore tends to a limit $r \geq 0$. If possible suppose $r > 0$. Then for given $\delta(r) > 0$, there exists a positive integer N such that for each $n \geq N$, we have

$$r \leq \psi(M(x_{2n}, x_{2n+1})) < r + \delta(r), \tag{3.3}$$

and $$r \leq \psi(d(y_{2n+2}, y_{2n+3})) < r + \delta(r) \tag{3.4}$$

or $$\psi(r) \leq \psi(r + \delta(r)) \tag{3.5}$$

But then, for each $n \geq N$, $\psi(r) \leq M(x_{2n+2}, x_{2n+1}) < \psi(r + \delta(r))$ which, by (iii) yields,

$\psi(\delta(Fx_{2n+2}, Gx_{2n+1})) < r$, implying thereby $d(y_{2n+2}, y_{2n+3}) < r$,

which is contradicting (3.4). Hence $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

We now show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists an $\varepsilon > 0$ and a Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, there exists an positive integer N such that

$$d(y_n, y_{n+1}) < \frac{\delta(\varepsilon)}{6}, \text{ whenever } n \geq N.$$

Let $n_i \geq N$ then there exists integer m_i satisfying $n_i < m_i < n_{i+1}$ such that

$$d(y_{n_i}, y_{m_i}) \geq \varepsilon + \frac{\delta(\varepsilon)}{3}.$$

If not then

$$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}}) + d(y_{n_{i+1}}, y_{n_{i+1}})$$

$$< \varepsilon + \frac{\delta(\varepsilon)}{3} + \frac{\delta(\varepsilon)}{6} = \varepsilon + \frac{2\delta(\varepsilon)}{3} < 2\varepsilon$$

a contradiction. Without loss of generality we can assume n_i to be odd. Let m_i be the smallest even integer satisfying

$$d(y_{n_i}, y_{m_i}) \geq \varepsilon + \frac{\delta(\varepsilon)}{3}.$$

Then $d(y_{n_i}, y_{m_{i-2}}) < \varepsilon + \frac{\delta(\varepsilon)}{3}$

and

$\varepsilon + \frac{\delta(\varepsilon)}{3} \leq d(y_{n_i}, y_{m_i})$

$\leq d(y_{n_i}, y_{m_{i-2}}) + d(y_{m_{i-2}}, y_{m_{i-1}}) + d(y_{m_{i-1}}, y_{m_i}) < \varepsilon + \frac{\delta(\varepsilon)}{3}$

$$+ \frac{\delta(\varepsilon)}{6} + \frac{\delta(\varepsilon)}{6} = \varepsilon + \frac{2\delta(\varepsilon)}{3} \qquad (3.6)$$

Also, $d(y_{m_i}, y_{n_i}) \leq M(x_{m_i}, x_{n_i}) < \varepsilon + \frac{2\delta(\varepsilon)}{3}$

$+ \frac{\delta(\varepsilon)}{6} + \frac{1}{2}\frac{\delta(\varepsilon)}{6} < \varepsilon + \delta(\varepsilon)$.

In view of (iii), this yields

$\delta(Fx_m{}^{i}, Gx_n{}^{i}) < \varepsilon$, implying, thereby, $d(y_m{}^{i-1}, y_n{}^{i+1}) < \varepsilon$.
But then
$d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{m_{i+1}}) + d(y_{n_{i+1}}, y_{m_{i+1}}) + d(y_{m_{i+1}}, y_{m_i})$

$$< \frac{\delta(\varepsilon)}{6} + \varepsilon + \frac{\delta(\varepsilon)}{6} = \varepsilon + \frac{\delta(\varepsilon)}{3}$$

which is a contradiction since $d(y_{n_i}, y_{m_i}) \geq \varepsilon + \frac{\delta(\varepsilon)}{3}$. Therefore $\{y_n\}$, and hence any subsequence thereof, is a Cauchy sequence.

Suppose J(X) is complete. Since $\{y_{2n+1}\} = \{Jx_{2n+1}\}$ is a subsequence of $\{y_n\}$, by the above $\{Jx_{2n+1}\}$ is Cauchy and $Jx_{2n+1} \to p = Jv$ for some $v \in X$.

We now show $Ix_{2n+2} \to p$. For suppose $Ix_{2n+1} \to q$. Since $y_{2n+1} = Jx_{2n+1} \to p$ and $y_{2n+2} = Ix_{2n+2} \to q$ therefore, $d(y_{2n+1}, y_{2n+2}) \to d(p, q)$.

But $\{d(y_{2n+1}, y_{2n+2})\}$ is a subsequence of the strictly decreasing sequence $\{d(y_n, y_{n+1})\}$ which tends to the limit $r = 0$. Therefore $\{d(y_{2n+1}, y_{2n+2})\}$ tends to lim $r = 0$ and hence $d(q, p) = 0$ implying $q = p$. Thus $Ix_{2n+2} \to p$. Now using (iii), we have

$\varepsilon \leq M(x_{2n}, x_{2n+1}) = \psi(\max\{d(Ix_{2n}, Jx_{2n+1}), D(Ix_{2n}, Fx_{2n}),$

$$D(Jx_{2m+1}, Gx_{2m+1}), \frac{D(Ix_{2n}, Gx_{2n+1}) + D(Ix_{2n+1}, Fx_{2n})}{2}\}),$$

$= \psi(\max\{d(Ix_{2n}, p), d(Ix_{2n}, p), d(p, Ix_{2n}),$

$$\frac{d(Ix_{2n}, Ix_{2n+2}) + d(Ix_{2n+2}, p)}{2}\}),$$

$= \psi(d(Ix_{2n}, p)) < d(Ix_{2n}, p) < \varepsilon + \delta(\varepsilon)$

which implies $\psi(\delta(Fx_{2n}, p)) < d(Fx_{2n}, p)) < \varepsilon$.

Consequently, $\delta(Fx_{2n}, p) \to 0$ as $n \to \infty$. In the same manner, it follows that $\delta(Gx_{2n+1}, p) \to 0$ as $n \to \infty$. We now show $Gv = \{p\}$. For this in view of (iii), we have

$\psi(\delta(Fx_{2n}, Gv)) < \varepsilon$, whenever $\varepsilon \leq M(x_{2n}, v) < \varepsilon + \delta(\varepsilon)$

where

$M(x_{2n}, v) = \max\{d(Ix_{2n}, Jv), D(Fx_{2n}, Ix_{2n}), D(Jv, Gv),$

$$\frac{D(Ix_{2n}, Gv) + D(Iv, Fx_{2n})}{2}\}$$

which gives $Fx_{2n} = Gv = \{p\}$ (3.7)

Now, if $\{p\} \not\subset Gv$ i.e. $D(p, Gv) \neq 0$ then letting $n \to \infty$ in (3.6) and using (3.7) we get,

$$\delta(p, Gv) < D(p, Gv)$$

which is a contradiction. But If $\{p\} \subset Gv$ i.e., $D(p, Gv) = 0$, then clearly from (3.6) and (3.7) $\lim_{n\to\infty} \delta(Fx_{2n}, Gv) = 0$ and then it immediately implies that $\delta(p, Gv) = 0$. Hence

$Gv = \{p\} = \{Jv\}$ (3.8)

Since $G(X) \sqsubseteq I(X)$, there exists some $u \in X$ such that $Gv = \{Iu\}$. Hence $Gv = \{Jv\} = \{Iu\}$. We now show $Fu = \{Iu\}$. For this, first we prove $Iu \in Fu$. Suppose $Iu \notin Fu$ then $D(Iu, Fu) > 0$. Then we can find for a $\varepsilon > 0$, $\delta(\varepsilon) > 0$ in accordance with (iii) such that

$\varepsilon \leq M(u, v) = D(Iu, Fu) < \varepsilon + \delta(\varepsilon)$

which implies

$\delta(Fu, Gv) < \varepsilon$ i.e. $\delta(Fu, Iu) < \varepsilon$

while $\delta(Fu, Iu) \geq D(Fu, Iu) \geq \varepsilon$.

Therefore a contradiction arises. Hence $Iu \in Fu$.

But then $M(u, v) = 0$ which, by (iii), implies that

$$\text{diam } Fu = \text{diam } Gv = \text{diam } \{Iu\} = 0.$$

Therefore Fu is a singleton. Since $Iu \in Fu$ and Fu is a singleton, $Fu = \{Iu\}$. Hence

$$Gv = \{Jv\} = Fu = \{Iu\} = \{p\} \quad (3.9)$$

Since the pair {F, I} and {G, J} are weakly compatible,
$Fp = FIu = \{IFu\} = \{Ip\}$ and $Gp = GJu = \{JGu\} = \{Jp\}$.
From the above, it is clear that Fp and Gp are singletons and

$$\delta(Fp, Gp) = d(Ip, Jp). \quad (3.10)$$

We now show that $Ip = Jp$. For instance, suppose $Ip \neq Jp$ then $d(Ip, Jp) > 0$.
Taking $d(Ip, Jp) = \varepsilon > 0$. Then, for each $\delta(\varepsilon) > 0$, we have

$$\varepsilon \leq M(p, p) < \varepsilon + \delta(\varepsilon)$$

which, in view of (iii), yields $\delta(Fp, Gp) < \varepsilon$, a contradiction. Hence $Ip = Jp$ and therefore

$$Fp = Gp = \{Ip\} = \{Jp\}. \quad (3.11)$$

We now show $Fp = \{p\}$. For, suppose $Fp \neq \{p\}$. Then $\delta(Fp, p) = d(Ip, p) > 0$.
Taking $d(Ip, p) = \varepsilon > 0$, we have, for each $\delta(\varepsilon) > 0$,

$$\varepsilon \leq M(p, v) < \varepsilon + \delta(\varepsilon)$$

implying, by (iii), $\delta(Fp, Gv) < \varepsilon$ i.e. $\delta(Fp, p) < \varepsilon$ a contradiction.
Therefore $Fp = \{p\}$ and hence $Fp = Gp = \{Ip\} = \{Jp\} = \{p\}$.
Again let $q \in X$ be any point satisfying (3.10) (3.11), we have

$$Fq = Gq = \{Iq\} = \{Jq\} = \{q\}. \quad (3.12)$$

Suppose $q \neq p$ then $d(q, p) > 0$. Taking $\varepsilon = d(q, p) > 0$ we have
$\varepsilon \leq M(q, p) < \varepsilon + \delta(\varepsilon)$ for each $\delta(\varepsilon) > 0$.
Selecting $\delta(\varepsilon)$ in accordance with (iii), we get in view of (iii)
$\delta(Fq, Gp) < \varepsilon$ i.e. $d(q, p) < \varepsilon$
which is a contradiction. Hence $q = p$. So that in (3.12),

$$Fp = Gp = \{Ip\} = \{Jp\} = \{p\}.$$

Corollary 3.1: Let I be a self mapping of a metric space (X, d) and $f: X \rightarrow B(X)$ a set valued mapping satisfying

(i)* $f(X) \subset I(X)$
(ii)* {f, I} are weakly compatible
(iii)* given an $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
$\varepsilon \leq d(Ix, Iy) < \varepsilon + \delta(\varepsilon)$ implies $\delta(fx, fy) < \varepsilon$.
If I(X) is complete subspace of X, there exists a unique point $p \in X$ such that
$fp = \{Ip\} = \{p\}$.

Proof: Taking I = J and F = G = f in Theorem 3.1.

Corollary 3.2: Let (X, d) be a complete metric space and $f : X \to B(X)$ a set valued mapping satisfying (iii) given an $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
$\varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon)$ implies $\delta(fx, fy) < \varepsilon$.
Then f has a unique fixed point in X.

Proof: Taking I = identity mapping in Corollary (3.1).

4. REFERENCES

1. M.A. Ahmed, Common fixed point theorems for weakly compatible mappings, Rocky MJM.,33 (2003), 1189-1203.
2. B. S. Choudhury and N. Metiya, Multivalued and singlevalued fixed point results in partially ordered metric spaces, Arab Journal of Mathematical Sciences (2011) 17, 135–151
3. B. S. Choudhury, A common unique fixed point result in metric spaces involving generalised altering distances. Math Commun, 10(2005), 105–110.
4. B. Fisher, Common fixed point of four mappings, Bull. Inst. Math. Acad. Sci., 11 (1983), 103-113.
5. B. Fisher, Common fixed point theorems for commuting mappings, Bull. Inst Math. Acad. Sinica 9 (1981), 399-406.
6. B. Fisher, Common fixed points of mappings and set valued mappings, Rostock. Math. Kolloq. 18(1981), 69 – 77.
7. B. Fisher, and S. Sessa, Two common fixed point theorems for weakly commuting mappings .Period. M. Hungar., 20 (1989), 207-218.
8. J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets. Nonlinear Anal 2009;71:3403–10.
9. J. Jachymski, Equivalent conditions and the Meir—Keeler type theorems, J. Math. Anal. Appl. 194 (1995), 293—303.
10. K. Jha, R. P. Pant and S. L. Singh, Common fixed point for compatible mappings in metric spaces, Radovi M., 12 (2003), 107-114.
11. G. Jungck and B. E. Rhoades, Some fixed point theorems for compatible maps, Internat. J. Math. Math. Sci. 16 (1993), 417-428.
12. G. Jungck, Common fixed points for commuting and compatible maps on Compacta, Proc. Amer. Math. Soc. 103 (1988), 977-983.
13. G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, Indian JPAM. 16(1998), 227-238.
14. T. C. Lim, On characterizations of Meir-Keeler contractive maps, Nonlinear Anal. 46 (2001) 113—120.
15. M. Maiti and T. K. Pal, Generalization of two fixed point theorems, Bull. Calcutta Math. Soc. 70(1978), 57-61.
16. A. Meir and E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28(1969), 326-329.
17. R. P. Pant and K. Jha, A generalization of Meir Keeler type fixed point theorem for four mappings, Ultra - Science, 15 (1) (2003) 97-102.
18. R. P. Pant and K. Jha, A generalization of Meir Keeler type Common fixed point theorem for four mappings, J. natural and Physical Sciences 16(1-2)(2002) 77-84.
19. S. Park and B. E. Rhoades, Meir-Keeler type contractive condition, Math. Japonica, 26(1)(1981) 13-20.
20. J. H. N. Rao and K. P. R. Rao, Generalization of fixed point theorem of Meir-Keeler type, IJPAM., 16(1)(1985), 1249-1262.
21. Rashwan, R.A., and Ahmed, M.A., fixed points of single and set valued mappings, Kyungpongk Math. J. 38(1998), 29-37.

22. R. A. Rashwan and M. A. Ahmed, Common fixed points for □-compatible mappings, Southwest J. Pure Appl. Math., 1 (1996), 1-6.
23. A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), no. 5, 1435–1443.
24. B.E. Rhoades, Some theorems on weakly contractive maps. Nonlinear Anal 2001;47(4):2683–93.
25. B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72 (2010), 4508–4517.
26. T. Suzuki, Fixed point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces, Nonlinear Anal. 64 (2006) 971—978.

******

----

[1]*Associate Professor,*
Department of Mathematical Sciences and Computer Applications,
Bundelkhand University, Jhansi, India
rksaini03@yahoo.com