A STUDY ON SPECIAL TRIANGULAR ARRAYS

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***Abstract:** In this paper we study an extension of rectangular arrays which we call as polygonal arrays. We propose some new array generation operations and study their properties. Similar to sturmian arrays, in this paper we concentrate on triangular Fibonacci arrays which is a particular case of polygonal arrays. The connection between polygonal arrays and triangular Fibonacci arrays could be interesting since polygonal arrays are simple and important discrete structures that appear in several problems related to theoretical computer science and discrete mathematics. We discuss several properties of triangular Fibonacci arrays.*

***Keywords:** Catenation operation, Leg word, Palindrome, Polygonal array, Triangular Fibonacci array.*

1. INTRODUCTION

The study of the combinatorial properties of finite sequence of symbols over a finite set is a great interest with remarkable applications in various fields such as Algebra, Physics, Computer Science and Biology. Aldo de Luca introduced a combinatorial method for the analysis of finite words for the study of biological molecules[1],[2]. On the other hand the three volumes of Handbook of Formal Languages edited by Rozenberg and Salomaa [9] give a comprehensive account of most of the developments that have taken place in the theory of formal languages, which is now an area of fundamental importance in Theoretical Computer Science, besides having applications to many fields including picture generating models.

The study of two-dimensional languages is a research topic which recently acquainted some interest in different fields of mathematics and computer science. Since 1967, there has been an attempt to extend results and techniques used on string (one-dimensional) languages to two-dimensional case, defining the two-dimensional languages or picture languages [8]. Another field of research in which two-dimensional languages recently took relevance, is that of DNA computation. Especially, Fibonacci word has been studied for many years. The combinatorial properties of some word generation functions are of great interest in some aspects of mathematics and computer science, such as number theory, fractal geometry, pattern matching [3]. Fibonacci based constructions are currently used to model physical systems with aperiodic order such as quasicrystals. Crystal growth techniques have been used to grow Fibonacci layered crystals and study their light scattering properties. Motivated by the studies we have proposed a new kind of array called polygonal array. The main motivation for the introduction is that to generate triangular fibonacci arrays.

This paper is organized as follows: Section 2 contains the basic definitions and notations about two-dimensional languages. In particular we give the definitions of the most important classes of two-dimensional languages used in our work. Section 3 discusses polygonal arrays and catenation operation between them. In section 4 a triangular Δ operation is defined which produce triangular fibonacci arrays. Some results on triangular fibonacci arrays are proved.

2. DEFINITIONS AND PRELIMINARY RESULTS

Let Σ be a finite alphabet. The set of all words over Σ is denoted by Σ^*. The empty word is denoted by λ. We write $\Sigma^+ = \Sigma^* - \{\lambda\}$. An infinite word w over a finite alphabet Σ is a mapping from positive integers into Σ. We write every infinite word w as $w = a_1 a_2 \dots a_i \dots$ where $a_i \in \Sigma$. For more details reader can refer [5],[7].

We now present the notion of picture and picture language. A picture p over Σ is a rectangular array of symbols of Σ. The set of all pictures over Σ is denoted by Σ^{**}. Given
$p \in \Sigma^{**}$ $\ell_r(p)$ and $\ell_c(p)$ denote the number of row and columns, respectively, of p. A picture language L is a subset of Σ^{**}. For more details reader can refer [4],[9].

3. POLYGONAL ARRAYS

This section introduces a new kind of arrays called polygonal arrays which are arrays of any shape. We will first introduce some definitions about two-dimensional polygonal arrays and polygonal languages. Here we consider arrays with connected property.

Definition 1 (Neighboring or Adjacent Symbols). Two symbols are called neighbours or adjacent if they are in positions (i, j) and (h, k) such that $|i-h| + |j-k| = 1$.

Definition 2 (Connected Array). An array A over Σ is called connected if it contains non-blank symbols and for any two symbols there exists a sequence (A_0, A_1, …, A_n) called path such that A_i is a neighbour of A_{i-1} where $1 \le i \le n$.

Definition 3. A two-dimensional array is a polygonal array if it satisfies one of the following conditions.
For $n \ge 2$, each n^{th} row contains equal or lesser number of entries than the $(n-1)^{th}$ row.
For $n \ge 2$, each n^{th} row contains equal or more number of entries than the $(n-1)^{th}$ row.

In this paper we take the bottom most row as the first row and the left most column as the first column like first quadrant XY of the two-dimensional plane. We now give the notation of polygonal arrays.

Definition 4. A polygonal array is called as a rightup polygonal array if it satisfies the first condition of Definition 3.

Definition 5. A polygonal array is called as a rightdown polygonal array if it satisfies the second condition of Definition 3.

We note that if a polygonal array has equal number of entries in all the rows then it is the standard rectangular array. Properties of such arrays are studied in [9]. Here we consider the rightup polygonal arrays and study their basic properties whereas the properties of other type follow immediately from rightup polygonal array. More properties are studied in [6].

A two-dimensional P-string over Σ is a two-dimensional polygonal array of elements of Σ. The set of all two-dimensional P-strings over Σ is denoted by Σ^{PP}. A two-dimensional P language over Σ is a subset of Σ^{PP}. We note that $\Sigma^{**} \subseteq \Sigma^{PP}$.

Given a P-picture $p \in \Sigma^{PP}$, let $\ell_r(p)$ denote the number of rows of p, $\ell_c(p)$ denote the number of columns of p and $\hat{p}_r = [P_1, P_2, \ldots]$; $\hat{p}^c = [P^1, P^2, \ldots]$ denote the number of entries in each row and in each column respectively. That is P_i denotes the number of entries in the i^{th} row and P^i denotes the number of entries in the i^{th} column. The ordered triple $(\ell_r(p), \ell_c(p), \hat{p}_r)$ or $(\ell_r(p), \ell_c(p), \hat{p}^c)$ is called the shape of p and for rectangle pictures we simply call size and is denoted by $(\ell_r(p), \ell_c(p))$. For square arrays we note that $\hat{p}_r = \hat{p}^c$. We call $\ell_r(p)$ as the base of p and $\ell_c(p)$ as the height of p. It is clear that a polygonal array of size $(m, n, \hat{p}_r)$ has minimum of $(m+n-1)$ entries and maximum of mn entries. Suppose that it has equal number of rows and columns and has $\frac{n(n+1)}{2}$ entries then it is in triangular shape. We note that for a triangular array $p \in \Sigma^{PP}$, $P_{i-1} - P_i = 1$. The following figure show the above types.

a	a a a a a	a
a	a a a a a	a a
a	a a a a a	a a a
a	a a a a a	a a a a
a a a a a	a a a a a	a a a a a
L shape	***□ shape***	***Δ shape***

Fig. 1.

The empty picture is the only picture of size (0, 0) and it will be denoted by λ. Pictures of size (0, n) or (n, 0) where $n > 0$ are not defined. The set of all P-pictures over Σ of size $(m, n, \hat{p}_r)$ with $m, n > 0$ will be indicated by $\Sigma_P^{m\times n}$. Furthermore, if $1 \le i \le \ell_r(p)$ and $1 \le j \le \ell_c(p)$. p(i, j) or equivalently p_{ij} denotes the symbol in p with coordinates (i, j).

Definition 6. Let $p \in \Sigma^{PP}$ of shape (m, n, $\hat{p}_r$). An array p′ of shape (m′, n′, $\hat{p}'_r$) is a subarray of p if m ≥ m′, n ≥ n′, $p_r \geq \hat{p}'_r$ and if there exist positive integers r, s such that $p'_{ij} = p'_{(i+r)(j+s)}$, 0 ≤ r ≤ m–m′, 0 ≤ s ≤ n–n′.

We now define some concatenation operation between P-pictures and two-dimensional P-languages. Let p and q be two pictures over an alphabet Σ of size (m, n, $\hat{p}_r$) and (m′, n′, $\hat{q}_r$), m, n, m′, n′ > 0 respectively. We simply denote a picture p of size (m, n, $\hat{p}_r$) as p = p(m, n, $\hat{p}_r$).

Definition 7. The column catenation of p and q (denoted by p $\oplus$ q) is a partial operation, defined only if m′ ≤ m and is given by

p $\oplus$ q = r(m, n+n′, $\hat{r}_r$)

where

$$\begin{aligned} \hat{r}_r &= \hat{p}_r + \hat{q}_r \\ &= (P_1, P_2, ..., P_m) + (Q_1, Q_2, ..., Q_{m'}) \\ &= (P_1 + Q_1, P_2 + Q_2, ..., P_{m'} + Q_{m'}, P_{m'+1}, ..., P_m) \end{aligned}$$

Similarly, the row concatenation of p and q (denoted by p $\ominus$ q) is a partial operation defined only if n′ ≤ n, and is given by

p $\ominus$ q = r(m+m′, n, $\hat{r}_r$)

where

$$\begin{aligned} \hat{r}^c &= \hat{p}^c + \hat{q}^c \\ &= (P^1 + Q^1, P^2 + Q^2, ..., P^{n'} + Q^{n'}, P^{n'+1}, ..., Q^n) \end{aligned}$$

We also note that p $\ominus$ q is possible if $\ell_r(p) \geq \ell_r(q)$ and p $\ominus$ q is possible if $\ell_c(p) \geq \ell_c(q)$.

4. CONSTRUCTION OF SPECIAL ARRAYS

A sequence of words $(f_n)_{n\geq 0}$ defined as follows: f_0 = b; f_1 = a and $f_{n+1} = f_n f_{n-1}$ for n ≥ 1
are referred to as the finite fibonacci words. Denote $g_n = |f_n|$, so that the numbers g_0, g_1, g_2, correspond to the fibonacci numbers 1, 1, 2, 3, … respectively.

In this section we construct a sequence of arrays F_0, F_1, F_2, … over a binary alphabet Σ = {a, b}. We start the construction with the pair of (1 × 1) size arrays b and a which we call as initial arrays and we denote it by F_0, F_1 respectively where $F_0 \neq F_1$.

We now derive the successive arrays by the application both row catenation and column catenation between two consecutive arrays with the property that

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$$F_{n+1} = (F_n \ominus G_n) \oplus F_{n-1}$$
$$= F_n \Delta^n F_{n-1}$$

$$\Delta^n = (\ominus\ G_n) \oplus\ ;\ G_n = (g_{ij});\ F_{i-1} = (f_{ij});$$

and $g_{ij} = f_{1j}\ \ \forall\ \ i, j$, G_n is a array of size $(\ell_r(F_{n-1}), \ell_c(F_n)$ and every column of G_n is equal to the first column of F_{n-1}.

The array F_{n+1} is called immediate successor of F_n and F_n is their immediate predecessor. The following illustration shows the operation procedure at each step in a constructive way. By definition

$$F_2 = F_1 \Delta^1 F_0$$
$$= (F_1 \ominus G_1) \oplus\ F_0$$

$$F_3 = F_2 \Delta^2 F_1$$
$$= (F_2 \ominus\ G_2) \oplus F_1$$

The above operations produce arrays of the following type:

$$F_2 = F_1 \Delta^1 F_0$$
$$= (F_1 \ominus G_1) \oplus F_0$$

$$F_3 = F_2 \Delta^2 F_1$$
$$= (F_2 \ominus\ G_2) \oplus F_1$$

Fig. 2.

Lemma 1. For $n \geq 2$, G_n satisfies $g_{ij} = g_{ji}$ for $1 \leq i \leq \ell_r(F_{n-1})$, $1 \leq j \leq \ell_c(F_n)$.

Proof. Since $g_{ij} = f_{i1}$ for $1 \leq i \leq \ell_r(F_{r-1})$, $1 \leq j \leq \ell_c(F_r)$ and since $\ell_r(F_{r-1}) < \ell_c(F_r)$ we note that $g_{ij} = f_{i1} = g_{ji}$. □

Lemma 2. The word associated with every row coincides with the word corresponding column from an arbitrary position.

Proof. The result holds for n = 0, 1, 2. Suppose that the result is true for F_r. By construction

$$\boldsymbol{F_{r+1} = (F_r \ominus G_r) \oplus F_{r-1}}$$

and from Lemma 1 the result holds for F_{r+1}. □

Let F_n be a $(n+1)^{th}$ array of the sequence of F_n, $n \geq 0$. We define height or base of F_n as the length of the word associated with the first row (or first column) of F_n. We call this word as leg word of the array and is denoted by $L(F_n)$ and the height as $|L(F_n)|$.

We see that in Figure 2, $|L(F_1)|$, $|L(F_2)|$, $|L(F_3)|$, $|L(F_4)|$ and $|L(F_5)|$ are 1, 1, 2, 3 and 5 respectively.

Result 1. (i) $L[F_n \ominus G_n) \obar F_{n-1}] = L[F_n \ominus G_n]\, L[F_{n-1}]$
(ii) $L[F_n \ominus G_n] = L[F_n]$

Lemma 3. Let $\{F_n\}$, $n \geq 0$ be a sequence of arrays formed by Δ operation. Then for each n the leg word of F_n is a finite fibonacci word f_n.

Proof. We prove by induction. The leg words of F_n, for n = 0, 1, 2, 3, 4 are b, a, ab, aba, abaab respectively. But we know that these words are finite fibonacci words. Assume that for a finite r, the base word of F_r is a finite fibonacci word. By construction

$$F_{r+1} = F_r \, \Delta^r \, F_{r-1}$$

and by Result 1

$$\begin{aligned} L[F_{r+1}] &= L[F_r]L[F_{r-1}] \\ &= f_r f_{r-1}. \end{aligned}$$

This completes the proof. □

Corollary 1. Let $\{F_n\}$, $n \geq 0$ be a sequence of arrays formed by Δ operation. Then the height of successive arrays form the fibonacci numbers. More over $|L(F_n)| = |f_n| = g_n$ where f_n is finite fibonacci word and g_n is the fibonacci number.

Proof. By definition of height of an array, $|L(F_n)|$ is the length of the leg word. Therefore

$$\begin{aligned} L(F_n) &= |f_{n-1}| + |f_{n-2}| \\ &= g_{n-1} + g_{n-2} \\ &= g_n \end{aligned}$$

Therefore the heights of $\{F_n\}$, $n \geq 0$ are 1, 1, 2, 3, 5, …, which is a sequence of fibonacci numbers $g_0, g_1, g_2, \ldots$ respectively. □

Lemma 4. Let $\{F_n\}$, $n \geq 4$ be a sequence of arrays formed by Δ operation. Then for $n \geq 4$ in each F_n a 3×3 neighbourhood of a cell is of the form $a_i \in \Sigma = \{a, b\}$.

$$\begin{matrix} a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \end{matrix}$$

Fig. 3.

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Proof. The proof follows from Lemma 2. □

Corollary 2. Let $\{F_n\}$, $n \geq 0$ be a sequence of arrays formed by Δ operation. Then the number of columns in successive arrays of G_n follows the sequence of fibonacci numbers.

Proof. G_n is of size $(\ell_r(F_{n-1}), \ell_c(F_n))$ but $\ell_c(F_n) = |L(F_n)| = g_n$. This complete the proof. □

Corollary 3. The Δ operation of two successive arrays differ only by main diagonal and by sub diagonal.

Proof. To prove the corollary we have to show that
$F_{n+1}\ \Delta^{n+1}\ F_n$ and $F_n\ \Delta^n\ F_{n+1}$ differ only by main and sub diagonals. Since the fibonacci words differ only by last two letters and hence the result follows from Lemma 3. □

The arrays are formed by repeated Δ operation in the same way that fibonacci word is formed by repeated concatenation and the fibonacci numbers are formed by repeated addition. We call terms of sequence of arrays as triangular fibonacci arrays.

Definition 9. Let $A = (a_{ij})$ be an array over an alphabet. A word associated with a_{n1} $a_{n2} \ldots a_{nn-1}\ a_{nn}\ a_{n-1n} \ldots a_{1n}$ is called a right angled word.

Definition 10. Let $A = (a_{ij})$ be an array over the alphabet. If $a_{ij} = a_{m+1-i\ n+1-j}$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Then A is said to be a palindrome array.

Definition 11. Let $A = (a_{ij})$ be a triangular array over the alphabet. If $a_{ij} = a_{n+1-i\ n+1-j}$ for $i+j = n+1$. Then A is said to be a triangular palindrome array.

Lemma 5. Let F_n be a triangular fibonacci array. Then for $n \geq 2$ the right angled words of each F_n are palindrome words. In fact this is true for every triangular and rectangular subarray of F_n.

Proof. The proof follows if we prove the result for any triangular subarray of F_n. Consider one such subarray A of height h. Then A has rectangular words $w_1, w_2, \ldots$ and
$|w_1| = 1$; $|w_2| = 3$; $|w_3| = 5$. Consider one such word w_r of length $r < h-1$. We note that $w_1 = w_r$; $w_2 = w_{r-1}$; and so on. Hence w is a palindrome word. □

Theorem 1. Let F_n $(n \geq 2)$ be a triangular fibonacci array. Then F_n is a palindrome triangular array.

Proof. The proof is immediate from Lemma 1 and Definition 11.

5. CONCLUSION

In this paper we gave an idea of polygonal arrays and studied catenation operation between such arrays. We introduced array generation operation and obtained several properties of Triangular Fibonacci arrays. Future work includes the study of more combinatorial properties of array generation functions and seek new applications of generating functions. We also aim to study combinatorial properties of infinite Triangular Fibonacci array.

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