REFLECTIONS ON GÖDEL, AND RELATED TOPICS

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Abstract: This paper discusses a serious discrepancy between recursive function theory and Gödel's first incompleteness theorem. According to recursive function theory a recursive predicate is decidable. However the first incompleteness theorem states that there is an undecidable recursive predicate. This paper makes it clear that there is a crucial error in the proof of the first incompleteness theorem.

1) It has been thought that a diagonal sequence function (formula) determined by all finite length unary functions (formulas) is a finite length one. We prove that it is an essentially infinite length function (formula).

2) Gödel used the diagonal sequence function (formula) defined by all substitution functions (formulas) and regarded it as a recursive function. However it is not recursive. Therefore his proof is incorrect.

3) Many authors of the books on Gödel's theory derived the first incompleteness theorem using the diagonal theorem. We prove that the diagonal theorem does not hold.

4) There are several theorems based on the assumptions that the diagonal sequence function (formula) is finite length and the diagonal theorem holds. Among them there are Tarski's theorem, the recursion theorem and the fixed point theorem. These theorems must be reexamined.

Keywords: diagonal theorem, first incompleteness theorem, Gödel, recursive function theory.

1. INTRODUCTION

In 1931 Gödel [1] showed that there exists an unprovable and unrefutable recursive predicate. This astonishing claim is called the first incompleteness theorem. However this theorem contradicts recursive function theory. According to recursive function theory a recursive predicate is decidable. Strange to say, this inconsistency has not been discussed. This paper discusses the discrepancy between recursion theory and Gödel's theory and states that Gödel's proof of the first incompleteness theorem includes serious errors.

First of all we shall review briefly predicate logic and recursion theory. The Gödel number of a function (formula) is irrelevant to the computability of a function (the decidability of a formula). In order to harmonize the length of a function (formula) with recursion theory, we define the length of a function and that of a formula. Using this measure we can show that the length of a recursive function (predicate) is finite length, and that of a well formed form but not recursive is essentially infinite length.

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A diagonal sequence function (formula) based on all finite length unary functions (formulas) is defined, and it is proved that the diagonal sequence function (formula) is an essentially infinite length function (formula). Based on this fact we show that the diagonal sequence function (predicate) defined by substitution functions (predicates) is an essentially infinitely long expression. This function (predicate) is not recursive, though Gödel supposed that the function (predicate) is recursive. Therefore there is a serious error in Gödel's proof of the first incompleteness theorem.

Almost all books on Gödel's theory derived the incompleteness theorem using the diagonal theorem. However we can show that there does not exist the diagonal theorem. Furthermore the paper pointed out that there are questions on the proofs of some theorems such as Tarski's theorem, the recursion theorem, and the fixed point theorem.

2. PRELIMINARIES

We shall briefly review predicate logic and recursion theory. The predicate logic for an arithmetic with addition and multiplication follows Shoenfield [3], but the classification of symbols is slightly modified.

2.1. Predicate logic

Definition 1. (Symbols used in the predicate logic for an arithmetic) The symbols of the predicate logic for the arithmetic are defined as follows. (a1) individual constants (a, b, c, …), (a2) variables (x, y, z, …), (a3) function symbols (+, $\cdot$), (a4) a predicate symbol (=), (a5) logical symbols 1 ($\neg$,$\vee$), (a6) a logical symbol 2 ($\exists$), (a7) subsidiary symbols ((,), comma). $\exists$ is called an existential quantifier.

Let A and B be sets of collections of objects. A mapping from the set of n-tuples in A to B is called an n-ary function from A to B. A subset of the set of n-tuples in A is called an n-ary predicate in A. An occurrence of variable x in predicate A is bound in A, if it occurs in a part of A of the form $\exists$ x A , otherwise it is free in A.

Definition 2. (Terms for an arithmetic) (b1) An individual constant is a term. (b2) A variable is a term. (b3) If $t_1, t_2 \ldots, t_n$ are terms and f^n is an n-ary function, $f^n(t_1, t_2, \ldots, t_n)$ is a term. (b4) Let $\{t_1, t_2, t_3, \ldots\}$ be an infinite set of terms. Define $T_1 = t_1 + t_2 + t_3 + \ldots$ and $T_2 = t_1 \cdot t_2 \cdot t_3 \ldots$. T_1 and T_2 are terms.

Definition 3. (Formulas) (c1) If $t_1, t_2, \ldots, t_n$ are terms and P is an n-ary predicate, $P(t_1, t_2, \ldots, t_n)$ is a formula. (c2) If A and B are formulas, $A \vee B$ and $\neg$ A are formulas. (c3) Let $\{ A_1, A_2, A_3, \ldots \}$ be an infinite set of formulas. Define $R = A_1 \vee A_2 \vee A_3 \vee \ldots$. R is a formula. (c4) If C(x) is a formula and x is a free variable, $\exists$ x C(x) is a formula.

Formulas $A \rightarrow B$, $A \wedge B$, $A \leftrightarrow B$ and $\forall x A(x)$ are the abbreviations of $\neg A \vee B$, $\neg(A \rightarrow \neg B)$, $((A \rightarrow B) \wedge (A \leftarrow B))$ and $\neg \exists x \neg A(x)$, respectively. Symbol $\forall$ is a universal quantifier. A formula without free variables is called a sentence.

An expression is a sequence of symbols. A proof is a finite sequence of one or more formulas such that each formula of the sequence is either an axiom or an immediate consequence of preceding formulas of the sequence. If A is the last formula in a proof P, P is said to be a proof of A. A is said to be provable or to be a theorem. Sometimes $\exists x C(x)$ represents finite numbers of Cs such that $C(x)$ $(x = 1, 2, \ldots, n)$. We call such $\exists x C(x)$ a finite existential formula. If $\exists x C(x)$ represents an infinite number of Cs such as $C(x)$ $(x = 1, 2, \ldots)$, we call $\exists x C(x)$ an infinite existential formula.

2.2. Recursion theory

2.2.1. Recursive functions and predicates

We shall describe a primitive recursive function (predicate), a recursive function (predicate) and a partial recursive function (predicate).

(d1) Initial functions

The initial functions are defined for natural numbers. Let $\bar{x}$ be $x_1, x_2, \ldots, x_n$.
(a) The zero function: $Z(x) = 0$ for all x.
(b) The successor function: $S(x) = x'$ for all x, where x' is the successor of x.
(c) The projection functions: $U_n^i(\bar{x}) = x_i$ for $i = 1, 2, \ldots, n$.

(d2) Composition

Let $h_1, h_2, \ldots, h_r$ be r n-ary functions $(r \geq 1, n \geq 0)$. Let g and f be a r-ary function and an n-ary function, respectively. Define f as follows.

$$f(\bar{x}) = g(h_1(\bar{x}), \ldots, h_r(\bar{x})). \quad (1)$$

(d3) Primitive recursion

Let g and h be an n-ary $(n \geq 0)$ function and an (n+2)-ary one, respectively. Define an (n+1)-ary function f as follows.

$$f(\bar{x}, 0) = g(\bar{x}).$$
$$f(\bar{x}, y') = h(\bar{x}, y, f(\bar{x}, y)). \quad (2)$$

(d4) μ operator

Function $g(\bar{x}, y)$ is called regular, if there is a natural number y such that $g(\bar{x}, y) = 0$ for any $\bar{x}$. μ operator is to find the least y for a regular function. Assume that g is an (n+1) -ary regular function. Define an n-ary function f as follows.

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$$f(\bar{x}) = \mu y(g(\bar{x}, y) = 0). \quad (3)$$

Definition 4. (Primitive recursive functions (predicate)) A function is primitive recursive, if it can be obtained by finite applications of (d2) and (d3) beginning with initial functions (d1). If P is an n-ary predicate, we define an n-ary function r_P such that $r_{P(a)} = 0$, if P(a), and $r_{P(a)} = 1$, if $\neg$ P(a). We call r_P the representing function of P. P is primitive recursive, if r_P is primitive recursive.

Definition 5. (A general recursive function (predicate)) A function is general recursive, if it can be obtained by finite applications of (d2) ~ (d4) beginning with initial functions (d1). If the representing function of a predicate is general recursive, the predicate is general recursive.

If $f(\bar{x})$ and $g(\bar{x})$ have the same domain and the same value, we shall write

$$f(\bar{x}) = g(\bar{x}). \quad (4)$$

Assume that $h_k(\bar{x})$ $(k = 1, \ldots, r)$ are defined and their values are y_k $(k = 1, \ldots, r)$. $g(h_1(\bar{x}), \ldots, h_r(\bar{x}))$ is defined, iff

$g(y_1, \ldots, y_r)$ is defined. This is called the composition in the weak sense. Modify (d1) ~ (d4) to define a partial recursive function.

(d1') Initial functions

The initial functions are partial functions.

(d2') Composition

Composition (d2) is done between partial functions in the weak sense.

(d3') Primitive recursion

Primitive recursion (d3) is done between partial functions in the weak sense.

(d4') μ-operator

μ operator (d4) is defined under the condition that (3) is defined iff $g(\bar{x}, y)$ is defined.

Definition 6. (A partial recursive function (predicate)) A function is partial recursive, if it can be obtained by finite applications of (d2') ~ (d4') beginning with initial functions (d1'). If the representing function of a predicate is partial recursive, the predicate is partial recursive.

Definition 7. (A recursively enumerable predicate) A predicate P(x) is recursively enumerable, if there is a recursive predicate Q(x, y) such that $P(x) \leftrightarrow \exists y\, Q(x, y)$.

2.2.2. The length of a function (predicate)

We shall define the length of a function (predicate) that has the following property. If a function (predicate) is recursive, the length of it is finite. If a function (predicate) is a well formed form but not recursive, the length is infinite.

Definition 8. (The symbols of the predicate logic for recursive function theory and the basic symbols) The symbols of the predicate logic for the theory of computation are defined as follows. Let (ek) be (ak) in Definition 1, where k = 1, 2, 4, …, 7. Let (e3) be function symbols (Z, S, U, +, ·, μ). The symbols of (e1) ~ (e5) and (e7) are called the basic symbols for recursive function theory.

Definition 9. (The length of a function (predicate)) The length of a function (predicate) is defined as the number of the basic symbols in the function (predicate). If a function (predicate) consists of a finite number of basic symbols, it is a finite length function (predicate). If a function (predicate) is not a finite length function (predicate), it is an infinite length function (predicate). The length of a term (a formula and a predicate) are similarly defined. If a predicate with quantifiers is proved to be recursive, it is a finite length predicate. If it cannot, it is an infinite length predicate. Note that we sometimes define function symbols and predicate symbols that are not in the basic symbols. In this paper we assume that defined function symbols and defined predicate symbols are rewritten using the basic symbols.

Terms T_1 and T_2 in Definition 2 are infinite length terms. Formula R in Definition 3 is an infinite length formula. Let Q be a finite length formula without infinite existential formulas. Assume that $\exists xC(x)$ is an infinite existential formula and there is an axiom or a theorem such that $\exists xC(x) \rightarrow Q$. $\exists xC(x)$ can be converted to a finite length formula.

Definition 10. (An essentially infinite length function (predicate)) If an infinite length function cannot be transformed into a finite one with any efforts, the function is called an essentially infinite length function . An essentially infinite length formula and that predicate are defined in the similar way.

Definition 11. (Terms for recursive function theory) Let (fk) be (bk) in Definition 2, where k = 1, 2, 3, 4. The set of n-ary functions includes the initial functions, the functions defined by composition and those by μ operator. (f5) If $t_1, t_2, \ldots, t_n$ are terms and $f^{(n+1)}(x_1, x_2, \ldots, x_n, y)$ is an (n+1)-ary function, $f^{(n+1)}(t_1, t_2, \ldots, t_n, y)$ is a term, where $f^{(n+1)}(t_1, t_2, \ldots, t_n, y)$ is defined by primitive recursion and y = 0, 1, ….

Definition 12. (wfts, wffs, Wt, Wf) The terms generated by Definition 11 are called well formed terms (wfts, in abbreviation). The formulas generated by Definition 3 applying terms defined by Definition 11 are called well formed formulas (wffs, in abbreviation). Let Wt be the set of wfts and Wf be that of wffs. Wt includes infinite length terms and Wf does infinite length formulas.

Remark 2-1: There are mathematicians who believe that Gödel's logical system [1] does not generate infinite length terms and infinite length predicates. If we see only the definition of the system, it seems to be true. However, as we see later, function Sb(x,v,x) and predicate Bew(y) in Gödel's paper are infinite length.

We shall confirm that the computation of a primitive recursive function and that of a general recursive function terminate in finite operations. The initial functions Z(x) (= 0) and S(x) (= x') can be computed, if x is given. $U_n^i(\bar{x})$ (= x_i) is obtained, if $\bar{x}$ and i are given. If $\bar{x}$, h_1, h_2, **…**, h_r, g are given, a function $f(\bar{x})$ defined by composition can be computed.

Consider a primitive recursion. Assume that x and y are given. f(x, y) is computed in the following way.

f(x, 0) = g(x).

f(x, 1) = h(x, 0, f(x, 0)) = h(x, 0, g(x)).

f(x, 2) = h(x, 1, f(x, 1)).

f(x, y) = h(x, y-1, f(x, y-1)). (5)

The computation of a primitive recursive function terminates in finite operations, if x and y are finite.

A function obtained by finite applications of compositions and primitive recursions beginning with initial functions is a function with finite symbols. Therefore a primitive recursive function is a finite length function. If r_P is primitive recursive, P is a finite length predicate.

Remark 2-2. Both a primitive recursive function and a primitive recursive predicate are finite length.

Regular function $g(\bar{x}, y)$ has y_0 such that $g(\bar{x}, y_0) = 0$, where y_0 is finite. Even if y_0 is not known, y_0 can be obtained by finite times computations of $g(\bar{x}, y)$ for y = 0, 1, **…**. Define p(z) such that if z = 0, p(z) = 1, and otherwise, p(z) = 0. The function $\mu y[g(\bar{x},y) = 0]$ is expressed in the following way.

$\mu y[g(\bar{x},y) = 0]$

$= 0 \cdot p(g(\bar{x},0)) + 1 \cdot p(g(\bar{x}, 1)) + \ldots + y_0 \cdot p(g(\bar{x}, y_0))$. (6)

The function $\mu y[g(\bar{x}, y) = 0]$ is a finite length function. From Remark 2-2 and (6), a recursive function is finite length. If r_P is recursive, predicate P is finite length.

Remark 2-3. Both a recursive function and a recursive predicate are finite length.

2.2.3. Well formed recursive functions and predicates

The followings are important and widely accepted understanding of a recursive function.

(g1) A recursive function is a computable function.

(g2) A computable function is a recursive function.

The latter is called Church's thesis. From Remark 2-3 a recursive function is a finite length function.

Note that we study a function in Wt. That is, "a function" of "a finite length function" is "a recursive function". We have the following.
(g3) A recursive function is a finite length function in Wt, and vice versa.

From (g2) and (g3), we have the following.
(g4) A computable function is a finite length function in Wt, and vice versa.

We shall discuss a formula in Wf. Replacing a function with a predicate, we have the similar statements.
(h1) A recursive predicate is a decidable predicate.
(h2) A decidable predicate is a recursive predicate.
(h3) A recursive predicate is a finite length predicate in Wf, and vice versa.
(h4) A decidable predicate is a finite length predicate in Wf, and vice versa.

Note that since the general recursion theory includes Peano arithmetic, (h4) holds in the arithmetic. From (h4) we have the following Remark that contradicts the first incompleteness theorem.

Remark 2-4. There is no finite length predicate in Wf that cannot be proved nor refuted.

Let A be any finite length recursive predicate. From (h4) A is decidable. $A \vee \neg A$ is tautology, provable, and decidable. Furthermore, $A \wedge \neg A$ is always false, unprovable and decidable. The arithmetic cannot derive $A \wedge \neg A$. Theory T is inconsistent if every predicate of T is a theorem of T; otherwise, T is consistent. Therefore the arithmetic is consistent. The consistency of the arithmetic is provable in the arithmetic. The following Remark denies the second incompleteness theorem.

Remark 2-5. The arithmetic is consistent. The consistency of the arithmetic is provable in it.

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2.2.4. Well formed non recursive functions and predicates

Note that "a function" is "a well formed function". Since we study functions in Wt, the negation of "a finite length function" is "an essentially infinite length function in Wt", and the negation of "a recursive function" is "a non recursive function in Wt". The negations of both (g1) and (g2) introduce to (i1).
(i1) A non recursive function in Wt is a non computable function in Wt, and vice versa.

By the negations of both (g3) and (g4), we have (i2) and (i3), respectively.
(i2) A non recursive function in Wt is an essentially infinite length function in Wt, and vice versa.
(i3) A non computable function in Wt is an essentially infinite length function in Wt, and vice versa.

From (i2) we have the following.
(i4) A recursive enumerable but not recursive function is an essentially infinite length function.

Note that "a formula" is "a well formed formula". Since we study formulas in Wf, the negation of "a finite length formula" is "an essentially infinite length formula", and the negation of "a recursive predicate" is "a non recursive predicate in Wf ". Similarly we have the followings.

(j1) A non recursive predicate in Wf is an undecidable predicate in Wf, and vice versa.

(j2) A non recursive predicate in Wf is an essentially infinite length predicate in Wf, and vice versa.

(j3) An undecidable predicate in Wf is an essentially infinite length predicate in Wf, and vice versa.

From (j2) we have the following.

(j4) A recursively enumerable but not recursive predicate is an essentially infinite length predicate.

Lemma 2-1. A non recursive function in Wt is an essentially infinite length function. A non recursive predicate in Wf is an essentially infinite length predicate.

2.2.5. Finitism and the concept of infinity

Another question is the relation between finitism and the concept of infinity. As it is well known, it is believed that metamathematics must stand on finitism. Some mathematicians believe that we should not use the concept of an infinite length function (formula) from the finitism point of view. This viewpoint seems to be reasonable. However, as we pointed out in Remark 2-1, there appear several essentially infinite length formulas in Gödel's paper. Therefore we cannot discuss Gödel's theory without the definition of an infinite length function (formula). In finitism the law of excluded middle must be carefully applied. [13](pp.1~25)

3. DIAGONAL SEQUENCES

Most books on Gödel's theory derived the first incompleteness theory using the diagonal theorem. This method is simpler than Gödel's method and easy to understand. In this section we shall discuss the diagonal theorem and show that the theorem does not hold.

3.1. A diagonal sequence formula

Enumerate all finite length unary formulas with a free variable u. Let them be

$$A_1(u), A_2(u), \dots , A_k(u) , \dots. \qquad (7)$$

Let S_A be the set of them. Consider the following two sequences such as

$$\neg A_1(1), \neg A_2(2), \dots, \neg A_k(k) , \dots. \qquad (8)$$

$$A_1(1), A_2(2), \dots, A_k(k) , \dots \qquad (9)$$

(8) and (9) are called the antidiagonal sequence and the diagonal sequence of (7), respectively.

Define formula $g_k(u)$ such that if u = k, $g_k(u)$ = true, otherwise $g_k(u)$ = false. Consider formulas I(u) and J(u) such as

$$I(u) = (g_1(u)\land\neg A_1(1))\lor(g_2(u)\land\neg A_2(2))\lor\dots . \qquad (10)$$

$$J(u) = (g_1(u)\land A_1(1))\lor(g_2(u)\land A_2(2))\lor\dots . \qquad (11)$$

(10) and (11) are the formula defined by the antidiagonal sequence and that by the diagonal sequence of (7), respectively. The former is called the antidiagonal sequence formula, and the latter the diagonal sequence formula. If I(u) is in S_A, it is a finite length formula. Since $I(k) \neq A_k(k)$ (k = 1, 2, …), then $I(u) \neq A_k(u)$ (k = 1, 2, …) and $I(u) \notin S_A$. Since $I(u) \notin S_A$, it is not a finite length formula. I(u) cannot be transformed to a finite length formula. Therefore, it is an essentially infinite length formula. Furthermore, we have

$$J(u) = \neg I(u). \qquad (12)$$

Note that J(u) can be written as $A_u(u)$.

Lemma 3-1. Let S_A be the set of all finite length unary formulas with a free variable u and each member of S_A be written as $A_k(u)$ (k = 1, 2, …). $A_u(u)$ is an essentially infinite length formula.

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Change the order of formulas in S_A. Let it be as follows.

$$A'_1(u), A'_2(u), \ldots, A'_k(u), \ldots \ . \tag{13}$$

Let I'(u) and J'(u) be the antidiagonal sequence formula and the diagonal sequence formula of (13), respectively. Define I' (u) and J' (u) as follows.

$$I'(u) = (g_1(u) \wedge \neg A'_1(1)) \vee (g_2(u) \wedge \neg A'_2(2)) \vee \ldots. \tag{14}$$

$$J'(u) = (g_1(u) \wedge A'_1(1)) \vee (g_2(u) \wedge A'_2(2)) \vee \ldots \ . \tag{15}$$

I'(u) and J'(u) are also essentially infinite length formulas. Note that there are infinitely many different sequences defined by all finite length unary formulas. For each order of formulas, there exist the antidiagonal sequence formula and the diagonal sequence formula. They are essentially infinite length sequences regardless of the order of formulas.

Remark 3-1. The antidiagonal sequence formula and the diagonal sequence formula are defined for each sequence of all finite length unary formulas. They are essentially infinite length formulas regardless of the order of formulas.

3.2. The diagonal sequence function

Enumerate all finite length unary functions with one free variable u. Let them be

$$B_1(u), B_2(u), \ldots, B_k(u), \ldots. \tag{16}$$

Let S_B be the set of them. Consider the following sequences.

$$B_1(1) + 1, B_2(2) + 1, \ldots, B_k(k) + 1, \ldots \ . \tag{17}$$

$$B_1(1), B_2(2), \ldots, B_k(k), \ldots \ . \tag{18}$$

(17) and (18) are called a modified diagonal sequence and the diagonal sequence of (13), respectively. Note that there are infinitely many modified diagonal sequences such as $B_k(k) + i$ ($k = 1, 2, \ldots; i = 1, 2, \ldots$). Define function $h_k(u)$ such that if $u = k$, $h_k(u) = 1$, otherwise $h_k(u) = 0$. Consider functions K(u) and L(u) such as

$$K(u) = h_1(u)\cdot(B_1(1) + 1) + h_2(u)\cdot(B_2(2) + 1) + \ldots. \tag{19}$$

$$L(u) = h_1(u)\cdot B_1(1) + h_2(u)\cdot B_2(2) + \ldots \ . \tag{20}$$

(19) and (20) are called a modified diagonal sequence function and the diagonal sequence function of (16), respectively. It is easy to see that K(u) is not in S_B and an essentially infinite length function. Note that

$$K(u) = L(u) + 1. \tag{21}$$

From (21) the diagonal sequence function L(u) is also an essentially infinite length function. Note that K(u) can be written as $B_u(u)$.

Lemma 3-2. Let S_B be the set of all finite length unary functions with free variable u and each member of S_B be written as $B_k(u)$ (k = 1, 2, …). $B_u(u)$ is an essentially infinite length function.

Remark 3-2. There are infinitely many modified diagonal sequences. Each modified diagonal sequence function and the diagonal sequence function are essentially infinite length functions regardless of the order of functions.

3.3. Unprovable predicates

In this subsection we shall describe an essentially infinite length function and that formula in Gödel's paper. Gödel defined function Sb(x, v, y) [1] (p.184, definition 31). However he regarded Sb(x, v, y) as a formula in the proofs of Theorem Ⅴ and Theorem Ⅵ [1] (pp.186~191).

In order to avoid confusion Maehara [9] used boldface and lightface, that is, x is the Gödel number of formula **x** which has a free variable **v**, v is the Gödel number of variable **v,** and y the Gödel number of term **y**. In this paper we follow Maehara, that is, Sb(x, v, y) is a function and **Sb**(**x**, **v**, **y**) is a formula. Gödel wrote that " Sb(x, v, y) is the notion Subst **a**(**v**,**b**) defined above". In another page [1] (p.177) he explained that "By Subst **a**(**v**,**b**) (where **a** stands for a formula, **v** for a variable, and **b** for a sign of the same type as **v**) we understand the formula that results from **a** if in **a** we replace **v**, wherever it is free, by **b**". Since **a** has a free variable **v**, **a** is **a**(**v**). Note that Subst **a**(**v**,**b**) is expressed as **Sb**(**a**, **v**, **b**). In footnote 20 in [1], Gödel wrote , " In case **v** does not occur in **a** as a free variable we put Subst **a**(**v**,**b**) = **a**. Note that "Subst" is a metamathematical sign." We shall rewrite Sb(x, v, y) with **Sb**(**x**, **v**, **y**), and treat **Sb**(**x**, **v**, **y**) as a formula. Z(n) is the numeral of number n [1](p.183, definition 17). In Sb(x, v, y), y is the Gödel number of term **y**. However he used the notation Sb(x, v, Z(y)). This sign method confuses a reader. We would like to use **Sb**(**x**, **v**, Z(y)) instead of Sb(x, v, Z(y)). If **x** is a formula with a free variable **v**, v is a free variable and y is the Gödel number, we state that **Sb**(**x**, **v**, Z(y)) is meaningful. Enumerate meaningful formulas.

$$\mathbf{Sb}(\mathbf{x_1}, \mathbf{v}, Z(x_1)), \mathbf{Sb}(\mathbf{x_2}, \mathbf{v}, Z(x_2)), \ldots, \mathbf{Sb}(\mathbf{x_k}, \mathbf{v}, Z(x_k)), \ldots \quad (22)$$

Note that $\mathbf{Sb}(\mathbf{x_k}, \mathbf{v}, Z(v)) = \mathbf{x_k}(\mathbf{v})$. Therefore $\mathbf{Sb}(\mathbf{A_k}, \mathbf{u}, Z(u)) = \mathbf{A_k}(\mathbf{u})$, where $\mathbf{A_k}(\mathbf{u})$ is the boldface expression of $A_k(u)$ in (7). Hence **Sb** can represent each of (7). Notice that a boldface expression for a formula is only used in this subsection 3.3. Rewrite them in the following way.

$$\mathbf{Sb_1}(1, v, 1), \mathbf{Sb_1}(2, v, 2), \ldots, \mathbf{Sb_1}(k, v, k), \ldots \quad (23)$$

$\mathbf{Sb_1}(x, v, x)$ is the diagonal sequence formula of { $\mathbf{Sb_1}(x, v, y)$ (x = 1, 2, …; y = 1, 2, …) }. From Lemma 3-1 $\mathbf{Sb_1}(x, v, x)$ is an essentially infinite length formula. So is **Sb**(**x**, **v**, Z(x)). From (j2) **Sb**(**x**, **v**, Z(x)) is not recursive.

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Lemma 3-3. Formula **Sb(x**, **v**, Z(x)) is an essentially infinite length formula and not recursive.

Let consider the case that Sb(x, v, y) is a function. We write Sb(x, v, y) instead of Sb(x, v, Z(y)). Since Z(y) is the numeral of y, it cannot be used in a function. Note that if at least one of {x, v, y} is not the Gödel number of {**x**, **v**, **y**}, respectively, Sb(x, v, y) is not defined. We would like to consider only meaningful functions. Let them be the followings.

$$Sb(x_1, v, x_1), Sb(x_2, v, x_2), \ldots, Sb(x_k, v, x_k), \quad (24)$$

Rewrite $Sb(x_k, v, x_k)$ with $Sb_2(k, m, k)$.

$$Sb_2(1, v, 1), Sb_2(2, v, 2), \ldots, Sb_2(k, v, k), \ldots \quad (25)$$

$Sb_2(x, v, x)$ is the diagonal sequence function of { $Sb_2(x, v, y)$ (x = 1, 2, …; y = 1, 2, …) }. From Lemma 3-2, Sb_2 (x, v, x) is an essentially infinite length function. So is Sb(x, v, x). From (i2) Sb(x, v, x) is not recursive.

Lemma 3-4. Function Sb(x, v, x) is an essentially infinite length function and not recursive.

Let us return to Gödel's notation. That is, we do not use boldface symbols hereafter. Let κ be the set of formulas of assumptions. $xB_\kappa y$ [1] (p.186, definition 45) indicates that y can be derived from κ and x is a proof of y. $Bew_\kappa(x)$ means that x is provable from κ. B is a primitive recursive formula. If a = Sb(y, m, Z(y)), $xB_\kappa a$ is an essentially infinite length formula. m is the Gödel number of a free variable in y.

Lemma 3-5. Formula $xB_\kappa Sb(y, m, Z(y))$ is an essentially infinite length formula and can neither be proved nor refuted.

Define the following formula [1] (p.186, definition 46). xBy is the case that κ is the empty set in $xB_\kappa y$.

$$Bew(y) = (\exists x)\ xBy \quad (26)$$

The right-hand side (26) means that there exists a proof x for y. Bew(y) means that y is provable. In order to understand the problem in the proof of the first incompleteness theorem, we shall quote Gödel 's words [1]. Some of the notations and definitions are not defined here, but we can refer to his paper for them.

We refer to Heijenoort's English translation [7] of Gödel's paper [1]. Gödel stated as follows.

[1] (p. 186) " Theorem V. For every recursive relation $R(x_1, \ldots, x_n)$ there exists an n-place RELATION SIGN r (with the FREE VARIABLES $u_1, \ldots, u_n$) such that for all n-tuples of numbers $(x_1, \ldots, x_n)$ we have

$R(x_1, \ldots, x_n)$

$$\rightarrow Bew[Sb(r, u_1, \ldots, u_n, Z(x_1), \ldots, Z(x_n))] \quad (27)$$

$\overline{R}(x_1, \ldots, x_n)$

$$\rightarrow Bew[Neg(Sb(r, u_1, \ldots, u_n, Z(x_1), \ldots, Z(x_n)))] \text{ "} \quad (28)$$

[1] (pp. 187-188) "We obviously have

$$(x)[Bew_\kappa(x) \sim x \text{ in } Flg(\kappa)] \quad (29)$$

and

$$(x)[Bew(x) \rightarrow Bew_\kappa(x)] \quad (30)$$

We now define the relation

$$Q(x,y) = Neg\{x\ B_\kappa[Sb(y,19, Z(y))]\} \quad (31)$$

Since $x\ B_\kappa y$ and Sb(y, 19, Z(y)) are recursive, so is Q(x, y). Therefore, by Theorem V and (30) there is a RELATION SIGN q (with the FREE VARIABLES 17 and 19) such that

$Neg[x\ B_\kappa[Sb(y, 19, Z(y))]]$

$$\rightarrow Bew_\kappa[Sb(q, (17, 19), (Z(x), Z(y))] \quad (32)$$

and

$x\ B_\kappa[Sb(y, 19, Z(y))]$

$$\rightarrow Bew_\kappa[Neg(Sb(q, (17, 19), (Z(x), Z(y)))] \text{ ".} \quad (33)$$

(x), $\overline{R}$ and Neg P indicate $\forall x$, $\neg R$ and $\neg P$, respectively. The numbers 17 and 19 are the Gödel numbers of certain free variables.

There is a serious mistake in the above description. Gödel claimed that Q(x, y) is recursive. That is, Gödel stated that " $x\ B_\kappa y$ and Sb(y, 19, Z(y)) are recursive, so is Q(x, y)". However, if we regard Sb(y, 19, Z(y)) as a formula, from Lemma 3-3 it is an essentially infinite length formula and not recursive. If we regard Sb(y, 19, Z(y)) as a function, from Lemma 3-4 it is an essentially infinite length function and not recursive. Therefore Q(x, y) is not recursive. Therefore Theorem V cannot be applied to Q(x, y). That is, there does not exist a relation sign q. Thus, Gödel's argument is wrong.

Remark 3-3. Gödel's proof of the first incompleteness theorem is not correct.

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4. DIAGONAL THEOREM

In many books on Gödel's theory the diagonal theorem is applied to derive the first incompleteness theorem. In this section we shall discuss the diagonal theorem and show that the theorem does not hold.

4.1 Diagonal theorem

We shall quote Smorynsky's explanation [8] (p.827) of the diagonal theorem.

"Let **T** be some fixed, but unspecified, consistent formal theory. Assume that the encoding is done in some fixed formal theory **S** and that **T** contains **S**. Define function sub(α,β) as follows.

$$\text{sub}(\ulcorner\varphi(x)\urcorner, \ulcorner t\urcorner) = \ulcorner\varphi(t)\urcorner, \tag{34}$$

where φ is any formula with one free variable, and $\ulcorner\varphi(x)\urcorner$ is the Gödel number for φ(x).

Diagonal theorem. Let φ(x) in the language of **T** have only the free variable indicated. Then there is a sentence ψ such that

$$\mathbf{S} \vdash \psi \leftrightarrow \varphi(\ulcorner\psi\urcorner). \tag{35}$$

Proof. Given φ(x), let

$$\theta(x) \leftrightarrow \varphi(\text{sub}(x,x)) \tag{36}$$

be the generalization of φ. Let

$$m = \ulcorner\theta(x)\urcorner, \psi = \theta(m). \tag{37}$$

Then we claim

$$\mathbf{S} \vdash \psi \leftrightarrow \varphi(\ulcorner\psi\urcorner). \tag{38}$$

For, in **S**, we see that

$$\Psi \leftrightarrow \theta(m) \leftrightarrow \varphi(\text{sub}(m, m)) \tag{39}$$

$$\leftrightarrow \varphi(\text{sub}(\ulcorner\theta(x)\urcorner, m)) \tag{40}$$

$$\leftrightarrow \varphi(\ulcorner\theta(m)\urcorner) \leftrightarrow \varphi(\ulcorner\psi\urcorner)." \tag{41}$$

Remark 4-1. Since variable v of Sb(x,v,x) in Lemma 3-4 is just a parameter, the lemma can be applied to sub(x,x) . Therefore sub(x,x) is an essentially infinite length function.

Then φ(sub(x,x)) is not defined. So is θ(x). The diagonal theorem has not been proved as it was intended in the beginning. Other proofs [14] (p.119), [15] (pp.172-173) have similar problems.

4.2. Does the diagonal theorem exist ?

Maehara [9] (pp.132-3) explained the substance of the diagonal theorem in the following way. Enumerate all formulas with one free variable x.

$$A_0(x), A_1(x), \ldots, A_k(x), \ldots \tag{42}$$

Consider the diagonal values.

$$A_0(0), A_1(1), \ldots, A_k(k), \ldots \tag{43}$$

Assume that an arbitrary univalent correspondence Φ between formulas is given. Let the following be the sequence of formulas obtained by (43) and Φ.

$$\Phi(A_k(k)) \ (k = 0, 1, \ldots). \tag{44}$$

Assume that formula G(x) represents (44). That is

$$\Phi(A_k(k)) = G(k) \ \ (k = 0, 1, \ldots). \tag{45}$$

G(x) is sure to be in (42). Let it be $A_n(x)$.

$$\Phi(A_k(k)) = A_n(k) \ \ (k=0,1, \ldots). \tag{46}$$

Put k = n. We have the following.

$$\Phi(A_n(n)) = A_n(n). \tag{47}$$

Let $A_n(n)$ be A.

$$\Phi(A) = A. \tag{48}$$

This is the substance of the diagonal theorem.

Remark4-2. Maehara [9] did not put any restrictions on formulas in (42). However each formula in (42) must be a finite length formula. As we have seen in Section 3, the diagonal sequence formula defined by all finite length unary formulas is an essentially infinite length formula. Since $A_x(x)$ is an essentially infinite length formula, $\Phi(A_x(x))$ is not defined. Hence, in general, there is no finite length G(x) such as (45) for any Φ. That is, we cannot find n of $A_n(x)$ in (46). This means the non-existence of the diagonal theorem.

5. RELATED TOPICS

In this section we shall point out some theorems derived from a diagonal sequence function (formula) and the diagonal theorem. These theorems were not proved correctly.

5.1. Tarski's theory

We shall refer Maehara's explanation [9] (pp.138-9).

Tarski's theorem: If a set of formulas K is consistent, there does not exist unary formula T(x) such that the following is proved from K

$$A \leftrightarrow T([A]), \tag{49}$$

where [A] is the term corresponding to the Gödel number $\ulcorner A \urcorner$ of formula A.

Proof: Assume that there exists such T(x) and show that K is inconsistent. Applying the diagonal theorem to unary formula $\neg$ T(x), we can prove that there exists sentence B such that

$$B \leftrightarrow \neg T([B]). \tag{50}$$

However if we substitute B for A in (49), the following is derived.

$$B \leftrightarrow T([B]). \tag{51}$$

(50) contradicts (51). Then K must be inconsistent. qed.

Remark 5-1. Since the diagonal theorem is applied in the proof, the proof is incorrect.

5.2. Recursion theorem

Many versions of recursion theorems are proposed. We shall quote Davis [11] (pp.98-9).

Recursion theorem: Let $g(e, x_1, \ldots, x_m)$ be a partially computable function of m+1 variables. Then there is a number e such that

$$\Phi_e^{(m)}(x_1, \ldots, x_m) = g(e, x_1, \ldots, x_m). \tag{52}$$

e is the Gödel number of Φ_e^m.

Note that usually $\tilde{=}$ is used to show "equality" between partial functions, but Davis uses =.

Proof. Consider the partially computable function

$$g(S_m^1(v, v), x_1, \ldots, x_m), \tag{53}$$

where S_m^1 is the function that occurs in the parameter theorem. Then we have for some number z_0,

$$g(S_m^1(v, v), x_1, \ldots, x_m)$$

$= \Phi^{(m+1)}(e, x_1, \ldots, x_m, v, z_0)$

$= \Phi^m(e, x_1, \ldots, x_m, S_m{}^1(v, z_0)))$,

where we have used the parameter theorem. Setting $v = z_0$ and $e = S_m{}^1(z_0, z_0)$, we have

$g(e,x_1, \ldots,x_m)$

$= \Phi^m(e, x_1, \ldots, x_m, e)$

$= \Phi_e{}^m(x_1, \ldots, x_m)$. qed.

Remark 5-2. $S_m{}^n$ is a primitive recursive function that appears in the parameter theorem (which has been called the iteration theorem or s-m-n theorem) [11] (p.95) such that

$$\Phi^{(m+n)}(x_1, \ldots, x_m, u_1, \ldots, u_n, y)$$

$$= \Phi^{(m)}(x_1, \ldots, x_m, S_m{}^n(u_1, \ldots, u_n, y)). \quad (54)$$

Put n = 1. $S_m{}^1(v, v)$ is the diagonal sequence function of all finite length unary functions of type $S_m{}^1(\alpha,\beta)$ $(\alpha, \beta = 1, 2, \ldots)$. Therefore $S_m{}^1(v, v)$ is an essentially infinite function. So the proof is incorrect.

5.3. Fixed point theorem

We shall quote Rogers [12](p.21, p.180). P_x is the set of instructions associated with the integer x in the fixed listing of all sets of instructions. x is called the Gödel number of P_x. $\varphi_x{}^{(k)}$ is the partial function of k variables determined by P_x. x is called Gödel number of $\varphi_x{}^{(k)}$. We shall drop the subscript (k) when its value is clear from context or when k = 1 .

The fixed point theorem: Let f be any recursive function; then there exists an n such that

$$\varphi_n = \varphi_{f(n)}. \quad (55)$$

(We call n is a fixed-point value for f.)

Proof. Let any u be given. Define a recursive function Ψ by the following instructions: to compute Ψ (x), first use P_u with input u; if and when this terminates and gives an output w, use P_w with input x; if and when this terminates, take its output as Ψ(x). We summarize this:

$$\Psi(x) = \varphi_{\varphi u(u)}(x), \quad \text{if } \varphi_u(u) \text{ is convergent;}$$

$$\text{divergent,} \quad \text{if } \varphi_u(u) \text{ is divergent.} \quad (56)$$

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The instructions for Ψ depend uniformly on u. Take $\hat{g}$ to be the recursive function which yields, from u, the Gödel number for these instructions for Ψ. Thus

$$\varphi_{\hat{g}(u)}(x) = \varphi_{\varphi u(u)}(x), \quad \text{if } \varphi_u(u) \text{ is convergent;}$$

$$\text{divergent, if } \varphi_u(u) \text{ is divergent.} \qquad (57)$$

Now let any recursive function f be given. Then $f\hat{g}$ is a recursive function. Let v be a Gödel number for $f\hat{g}$. Since $\varphi_v = f\hat{g}$ is total, $\varphi_v(v)$ is convergent. Hence putting v for u in the definition of $\hat{g}$, we have

$$\varphi_{\hat{g}\,(u)} = \varphi_{\varphi v(v)} = \varphi_{f\hat{g}\,(v)}. \qquad (58)$$

Thus n = $\hat{g}(v)$ is a fixed-point value, as desired. qed.

Remark 5-3. u is the Gödel number of $\varphi_u(x)$. $\varphi_u(u)$ is the diagonal sequence function of all finite length unary functions $\varphi_\alpha(x)$ (α= 1, 2, **…**). Therefore $\varphi_u(u)$ is an essentially infinite function. Hence $\varphi_{\varphi u(u)}(x)$ cannot be defined.

6. CONCLUDING REMARKS

It has been thought conventionally that the diagonal sequence function (formula) constructed by all finite length unary functions (formulas) is a finite length one. Based on this understanding the first incompleteness theorem and the diagonal theorem were proved. This paper made it clear that the diagonal sequence function (formula) is an essentially infinite length one. Therefore the first incompleteness theorem and the diagonal theorem were not proved correctly. This new knowledge coincides with Remark 2-4.

We did not discuss the second incompleteness theorem. The second theorem states that the consistency of an arithmetic cannot be proved in the system. If we pay attention to the facts that the second incompleteness theorem is proved based on the first one, the proof of the second theorem is incorrect. Remark 2-5 denies the second theorem. The theorem will probably not hold.

Though we did not state the case of partial recursive formulas (functions) , it can be discussed similarly [6].

Furthermore, the predicate to express the halting problem of a Turing machine is an essentially infinite length one [6].

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