SPECIAL PROPERTIES OF FIBONACCI ARRAY

N. Jansirani[1], V. Rajkumar Dare[2]

Abstract: In this paper we consider the concept of a palindrome. We define palindromic complexity, rich arrays and obtain relationship between palindromic complexity and sub array complexity. An interesting example of such a Fibonacci array is shown using the famous Fibonacci word which is a Sturmian word.

Keywords: Fibonacci word, Palindromic complexity, Rich array, Fibonacci array and sub word complexity.

1. INTRODUCTION

In 1984 a paper of Shechtman, Belech, Gratias and Cahn appeared, relating how they built an alloy of aluminum and manganese that has five fold symmetry. Since this is not possible for a crystal, and since this alloy is not a glass (an X-ray diffraction picture shows Bragg Peaks), the term quasi crystals was coined for such materials. This discovery led several physicists to begin studying one-dimensional structures based on the Fibonacci sequence f. Because the Fibonacci word is associated with Penrose tiling, theoretical physicists began to study the properties of f, which are not ultimately periodic but they are somewhere between periodicity (order) and chaos (disorder).They might correspond to (one dimensional) materials having physical properties between crystals and glasses, and might be a good theoretical model of one- dimensional quasi crystals.

The subject of crystallography is conceptual and imaginative. The bulk of experiments and theoretical methods available today permit the solution of practically any crystal structure. Therefore one can speck of an automation of the crystal structure analysis both from experimental and computational points of view [1].

In quasicrystals, one can find the fascinating mosaics of the Arabic world reproduced at the level of atoms: regular patterns that never repeat themselves. However, the configuration found in quasicrystals was considered impossible, and Daniel Shechtman had to fight a fierce battle against established science.

The Noble Prize in chemistry 2011 recognizes a breakthrough that has fundamentally altered how chemists conceive of solid matter. When scientists describe Shechtman's quasicrystals, they use a concept that comes from mathematics: the golden ratio. In quasicrystals, for instance. The ratio between atoms is related to the golden mean.

Finally, there is a purely combinatorial sufficient condition for singular continuous spectrum in terms of the arbitrarily long palindromic sub words occurring in the Fibonacci sequence. Motivated by this, in this paper we study the palindrome complexity of Fibonacci array [2].

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2. BASIC DEFINITIONS AND NOTATIONS

Let Σ be a finite alphabet. The set of all words over Σ is denoted by Σ^*. The empty word is denoted by λ. We write $\Sigma^+ = \Sigma^* - \{\lambda\}$.

An infinite word w over a finite alphabet Σ is a mapping from positive integers into Σ. We write $w = a_1 a_2 \ldots a_i \ldots$ where $a_i \in \Sigma$. The set of all infinite words over Σ is denoted Σ^ω. An infinite word w is ultimately periodic if $w = uv^\omega$.

An infinite word w over a binary alphabet $\{a, b\}$ which is not ultimately periodic and which is such that for any positive integer n, the number $g_x(n)$ of its factors of length n is minimal *i.e* $g_x(n) = n + 1$, is called a Sturmian word [1].The Fibonacci word f which is an important example of a Sturmian word is the fixed point of a morphism $\varphi : B^* \rightarrow B^*$ where $B = \{a,b\}$ and $\varphi(a) = ab$, $\varphi(b) = a$ *i.e.* $f = \varphi^\omega(a)$. In fact the first few symbols of the Fibonacci word is abaababaabaababaababa … .[8,9]. A finite word w is rich if and only if it has $|w|+1$ distinct palindromic factors including the empty word and an infinite word is rich if all of its factors are rich[5].

An $m \times n$ array $A = (a_{ij})_{mxn}$ over an alphabet Σ is a rectangular arrangement of symbols of Σ in m rows and in n columns. The size of the array A is the ordered pair (mxn). The set of all arrays over Σ is denoted by Σ^{**}. The empty array is also denoted by λ. We adopt the convention that for an $m \times n$ array $A = (a_{ij})_{mxn}$ the bottom most row is the first row and the left most column is the first column. Also we write $\Sigma^{++} = \Sigma^{**} - \{\lambda\}$. A factor or subarray of an array A is also an array which is a part of A.

An infinite array u has an infinite number of rows and infinite number of columns. The collection of all infinite arrays over Σ is denoted by $\Sigma^{\omega\omega}$. Row and column concatenation of arrays in Σ^{**} are partial operations. For row catenation of two arrays A and B , denoted by $A\theta B$, the number of columns in A and B should be equal and for column concatenation $A\phi B$ of A and B, the number of rows in A and B should be equal. The collection of all arrays with finite number of rows and an infinite number of columns is denoted by $\Sigma^{*\omega}$ and the collection of all arrays with an infinite number of rows and a finite number of columns is denoted by $\Sigma^{\omega *}$[3].

An array $w \in \Sigma^{\omega\omega}$ is said to be row ultimately periodic (Figure 1) if there exist two arrays u and v in $\Sigma^{*\omega}$ such that $w = u\theta v^{\theta\omega}$, $v^{\theta\omega}$ denotes an infinite number of row catenation of v with respect to v. An array $w \in \Sigma^{\omega\omega}$ is said to be column ultimately periodic (Figure 2) if there exist two arrays u and v of $\Sigma^{\omega *}$ such that $w = u \phi v^{\phi\omega}$, $v^{\phi\omega}$ denotes an infinite number of column catenation of v with respect to v Let $u = (u_{ij}) \in \Sigma^{**}$, $1 \le i \le m$, $1 \le j \le n$ [4],[7].

3. SUBWORD COMPLEXITY:

Definition. 1

The string or the word $u_{11}u_{12}u_{13} \ldots u_{1n}u_{2n}u_{3n} \ldots u_{mn}$ is called the right boundary of the array U and the string or the word $u_{11}u_{21}u_{31}u_{41} \ldots u_{m1}u_{m2}u_{m3} \ldots u_{mn}$ is called the left boundary of U .

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Note that the boundary of an array is formed by taking the right boundary and "joining" to it the reverse of the left boundary or taking the left boundary and "joining" to it the reverse of the right boundary. By "joining" we mean that a word *aub* joined to *bva* gives *aubv*. For an array $U \varepsilon \Sigma^{**}$ of size (m, n) the length of the right boundary or the left boundary is $m+n-1$.

Definition. 2

For any $U \in \Sigma^{**}$, the sub array complexity of U is the map g_U: N X N→N defined as

$$g_U(m, n) = \text{card}\,(S(U) \cap \Sigma^{m \times n})$$

where $g_A(m, n)$ counts the number of distinct sub arrays of A of size (m,n).

Definition. 3

An infinite array u which is neither row ultimately periodic nor column ultimately periodic is said to be a Sturmian array if $g_u(m, n) = m + n$ where $g_u(m, n)$ denotes the number of distinct subarrays or factors of u, of size (m, n).

Definition. 4

An array W is a finite sturmian array if there exists an infinite sturmian array U such that W Є F(U), where F(U), is the set of all (finite) factors of U. We denote by SFS, the set of all finite sturmian arrays.

Definition. 5

Let A be array of size (m,n). An extension of A is an array of size either (m,n+1) or (m+1,n). The two possible extensions are shown in the following

$$\begin{bmatrix} a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \end{bmatrix} \quad \begin{bmatrix} a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} & a_{mn+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} & a_{2n+1} \\ a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} & a_{1n+1} \end{bmatrix}$$

$$\begin{bmatrix} a_{m+11} & a_{m+22} & \cdot & \cdot & \cdot & a_{m+1n} \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \end{bmatrix}$$

Figure 1 : Extension of an Array

Example. 1

A good example of sturmian array is the Fibonacci array whose construction is shown in Figure 1 based on the Fibonacci word

a b a a b a b a a b …

. . b	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
a	b	.	.	.	.	.	.	.	.	.	.
a	a	b	.	.	.	.	.	.	.	.	.
b	a	a	b	.	.	.	.	.	.	.	.
a	b	a	a	b	.	.	.	.	.	.	.
b	a	b	a	a	b	.	.	.	.	.	.
a	b	a	b	a	a	b	.	.	.	.	.
a	a	b	a	b	a	a	b	.	.	.	.
b	a	a	b	a	b	a	a	b	.	.	.
a	b	a	a	b	a	b	a	a	b	.	.

Figure 2 : Fibonacci Array F

Since the Fibonacci word is a Sturmian word, it is clear from the construction that the array *F* formed in Figure 2 is a Sturmian array, which we call as a Fibonacci array.

We note that an extension of a sub array of size (m, n) in F there is only one new element. For instance, if an array W of size (m, n) is determined by a word w of length m+n-1 then extension of W is determined by either wa or wb.

Lemma. 1

Let be F a Fibonacci array. Then the following statement is true. There cannot be three or more consecutive occurrences of and there cannot be two or more consecutive occurrences of b either in any row or any column of F. It also guarantees that no sub arrays of the following type

$$\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \qquad \begin{bmatrix} b & b \\ b & b \end{bmatrix}$$

appear in the entire array (Figure 2).

Theorem. 1

Let F be a Fibonacci array. Then F satisfies the sub word complexity.

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Proof. We prove by induction. In fact, $g_u(1,1) = 2$ as there are only two letters *a, b*. Now the maximal value of $g_u(1,2)$ is 4 but we do not have the subarray *bb* in *F* and hence $g_u(1,2) = 3$. Similarly $g_u(2,1) = 3$. Likewise $g_u(2,2) = 4$, $g_u(1,3) = 4 = g_u(3,1) = 4$ and so on.

We now show that its complexity satisfies

$$g(m, n) = m + n \text{ for } m, n \geq 2 \qquad (1)$$

This is seen inductively by showing that [9], for each (m, n) there exists just one array W of size (m, n) such that both W θ A (W φ A) and W θ B (W φ B) are in SFS where A and B are extensions of W with a and b as new element respectively. Let us call such sub arrays special. For (1, 1), (2, 1), (1, 2) and (2, 2) the set of sub arrays are

a

b

a	b

b	a

a	a

a
b

b
a

a
a

a	b
b	a

b	a
a	b

b	b
a	a

Where

a

b	a

b
a

b	a
a	b

are special ones. Now consider a sub array W of size (m+1, n) with (m ,n) ≥2. If W ends with b, then, by the form of construction, W admits only the extension by a,

i.e , the a-extension. If W=X θ (W' θ A) with X whose end element is in {a , b}, then by induction hypothesis of the arrays W' θ A with $|W'| = (m,n-1)$, only one is special.

Therefore, we are done, when we show that of the arrays B θ(W' θ A) and C θ(W' θ A) with $|W'| \geq (1,1)$, only one can be factor of SFS where B and C has a and b has start element respectively.

Assume to the contrary that both of these words are in SFS. Then W' cannot start with b, therefore both B θ P θ (W" θ A) and C θ Q θ (W" θ A) , where P and Q has a and a as start element respectively for some W'', are factors of SFS. . By the form of the first of these ,W'' cannot start with a. But by the form of the second array, it cannot start with b either, since

b	b

$\notin$ SFS.

Hence, we have proved that F satisfies (1).

Definition. 6

An array $A = (a_{ij})$ over an alphabet for i≤m, j≤n is said to be symmetric if $a_{ij} = a_{ji} \; \forall (i,j)$.

Definition. 7

An array $A = (a_{ij})$ over an alphabet is said to be Pascal symmetric if $a_{ij} = a_{kl}$ for i+j = k+l, where i,j,k,l € N

Example. 2

$$\text{Let } A = \begin{pmatrix} 6 & 8 & 9 & 0 \\ 5 & 6 & 8 & 9 \\ 4 & 5 & 6 & 8 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

We note that Pascal symmetric implies symmetric but the converse need not be true. We show this by an example.

Example. 3

$$\text{Let } A = \begin{pmatrix} 6 & 8 & 9 & 0 \\ 5 & 2 & 1 & 9 \\ 4 & 7 & 2 & 8 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

We note that A is symmetric but not Pascal symmetric.

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Definition. 8

A finite array U is said to be a palindrome array if U is equal to the reversal of U with respect row and column. That is it is a 2D mirror image of U.

Let $U^R{}_C$ denotes column reversal of U and $U^R{}_R$ denotes row reversal of U and both operations applied together on a finite array is denoted by U^ We explain this concept by considering an array of size (5,5).

$$U = \begin{pmatrix} u_{51} & u_{52} & u_{53} & u_{54} & u_{55} \\ u_{41} & u_{42} & u_{43} & u_{44} & u_{45} \\ u_{31} & u_{32} & u_{33} & u_{34} & u_{35} \\ u_{21} & u_{22} & u_{23} & u_{24} & u_{25} \\ u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \end{pmatrix}$$

$$((U_C{}^R)_R)^R = \begin{pmatrix} u_{15} & u_{14} & u_{13} & u_{12} & u_{11} \\ u_{25} & u_{24} & u_{23} & u_{22} & u_{21} \\ u_{35} & u_{34} & u_{33} & u_{32} & u_{31} \\ u_{45} & u_{44} & u_{43} & u_{42} & u_{41} \\ u_{55} & u_{54} & u_{53} & u_{52} & u_{51} \end{pmatrix} = \hat{U}$$

if $U = \hat{U}$ then U is said to be palindrome.

Example. 4 An example of a palindrome array.

$$\text{Let } A = \begin{pmatrix} 6 & 5 & 4 & 3 \\ 8 & 2 & 7 & 1 \\ 1 & 7 & 2 & 8 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

Lemma. 2

An array $A = (a_{ij})$ be an array over an alphabet

If A is a Pascal symmetric then the right boundary and the left boundary of A are the same.

If A is a palindrome array then the right boundary (left boundary) is the reversal of the left boundary.

Remark.1

Any sub array of Pascal symmetric array is Pascal symmetric but this is not true for the case of palindrome property. (see example 3 and 4)

Definition. 9

A finite array is said to be boundary palindrome if its right boundary (left boundary) is a palindrome.

Example. 5

$$\text{Let } A = \begin{pmatrix} 2 & 5 & 1 & 3 \\ 5 & 2 & 5 & 4 \\ 1 & 5 & 2 & 5 \\ 3 & 4 & 5 & 2 \end{pmatrix}$$

Theorem. 2

Let $A = (a_{ij})$ be a Pascal symmetric array. Then A is a palindrome array If and only if A is a boundary palindrome.

Proof. The first part follows from lemma 2. Conversely if A is a lattice palindrome from definition of Pascal symmetric A is a palindrome array. (One can verify the theorem with example 3,4 and 5)

Theorem. 3

Let A be an infinite array over a binary alphabet. If A is a Fibonacci array then A is a Pascal symmetric array.

Proof. We prove the theorem for a sturmian array because Fibonacci array is an example of sturmian array. Let $A = (a_{ij})$ $i, j \in N$, $i \geq 1, j \geq 1$ be an infinite array. Assume that A sturmian is a array. By definition, the complexity function on A satisfies the minimality condition. Therefore, for any array of size (m,n) ,$g_A(m,n)= m+n$ is true. By definition there should be exactly m+1 distinct sub arrays of size (m ,1) , in A. The same result holds even if we consider a single row or single column. This implies that there should be some relation(ordering) between the rows and columns which in turn implies that any one of the row(column) generates the entire

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array. Without loss of generality let us assume that the first row generates the entire array. Let it be

X= $a_1\ a_2\ a_3$……… $a_n\ a_{n+1}$…………

By minimality condition , X is a sturmian word. Since every suffix of a sturmian is a distinct sturmian word and since to retain the minimality condition in A and in every row(column), we should not change the position. Hence

$a_{12} = a_2$. $a_{31} = a_3$, $a_{41} = a_4$........................... $a_{22} = a_3$, $a_{23} = a_4$…......................

The above description is shown in the figure1.

Definition. 10

A finite array is said to be rich if it is Pascal symmetric and its right boundary is rich.

An infinite array is said to be rich if all of its sub arrays are rich.

Definition. 11

Let W be an infinite array. Let P_W (m, n) denotes the number of distinct Pascal palindrome arrays of size (m, n).

Theorem. 4

An infinite array W is sturmian if and only if P_W (m, n) =1 when ever m+n-1 is even and P_W (m, n) =2 when ever m+n-1 is odd.

The main results of this paper is to give characterization of both rich palindromes and sturmian palindromes in terms of the palindromic complexity functions.

4. CONCLUSION

In this paper we have introduced the notion of a palindromic complexity, rich arrays and obtain relationship between palindromic complexity and sub array complexity array.

5. REFERENCES

1. J-P.Allouche and J. Shallit, *Automatic Sequences.*
2. Berstel and P. Séébold, Sturmian words. In: M. Lothaire, Editor, *Algebraic Combinatorics on Words, Encyclopedia of Mathematics and its Applications* vol.90, Cambridge University Press, UK (2002), 45–110.
3. V.R. Dare, K.G. Subramanian, D.G. Thomas and R. Siromoney "*Infinite arrays and Recognizability" Int. J. Pattern Recognition and Artificial Intelligence*, Vol. 14, **4**(2000) 525 – 536.
4. N. Jansirani, V. Rajkumar Dare, *"Combinatorics on Arrays", National Conference on Recent Advancement in Pure and Applied Mathematics ,Chennai, India(2009).*

5. Aldo. De Luca ,Amy Glen ,Luca Q Zamboni ,Rich, Sturmian and Trapezoidal words, *Theoretical Computer Science*, 407 (2008) 569 – 573. Elsevier.
6. Aldo. De Luca and Filippo Mignosi, "*Some Combinatorial Properties of sturmian words", Theoretical Computer Science*, **136** (1994) 361 – 385. Elsevier.
7. F. Mignosi, *" On the number of factors of Sturmian words"*, Theoretical Computer Science, **82** (1991) 71 – 84. Elsevier.
8. P.Séébold, *"Fibonacci morphisms and sturmian words"*, *Theoretical Computer Science,* **88** (1991) 365 – 384. Elsevier.
9. R. Siromoney, V.R. Dare and K.G. Subramanian, Infinite Arrays and Infinite Computations, *Theoretical Computer Science*, **24** (1983) 195-205.
10. G. Rozenberg, A. Salomaa (Eds.) "*Handbook of Formal Languages"*

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[1]N. Jansirani, Department of Mathematics, Queen Mary's College, Chennai 600 004, India.
njansirani@gmail.com
[2]V. Rajkumar Dare, Department of Mathematics, Madras Christian College, Chennai 600 059, India.