A STUDY ON THE MONOTONOCITY OF THE BIPLOAR FUZZY TRANSFORM

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Abstract:* *Fuzzy transforms developed by Irina Perfilieva is a novel, mathematically well founded soft computing tool with many applications. These techniques are based on mainly two transforms, direct fuzzy transform and inverse fuzzy transform. These two techniques are successfully applied by Irina Perfilieva for solving ordinary differential equation and many other problems. The concept of bipolar fuzzy transform has been introduced by the authors earlier. In this paper our aim is to study the inverse bipolar fuzzy transform and the monotonic property of the bipolar fuzzy transform and inverse bipolar fuzzy transform.

Keywords:* *Fuzzy transforms, Fuzzy partitions, Bipolar Fuzzy transforms.

1. INTRODUCTION

In classical Mathematics, various types of transforms are introduced (e.g. Laplace transform, Fourier transform, wavelet transform etc.) by various researchers. In 2001 Irina Perfilieva introduced fuzzy transform in her paper [2]. Latter on fuzzy transform is applied in to various fields, like numerical solution of differential equations, image processing, data mining etc in the papers [5, 10]. The fuzzy transform provides a relation between the space of continuous functions defined on a bounded domain of real line R and R^n .Similarly inverse fuzzy transform identified each vector of R^n with a continuous map. The central idea of the fuzzy transform is to partition the domain of the function by fuzzy sets.

Definition 1.1([5]). Let [a, b] be an interval of real numbers and $x_1 < x_2 < \ldots < x_n$ be fixed nodes within [a, b] such that $x_1 = a$, $x_n = b$ and $n \geq 2$ .We say that fuzzy sets $A_1, A_2, \ldots, A_n$ identified with their membership functions $A_1(x), A_2(x), \ldots, A_n(x)$ and defined on [a, b] form a fuzzy partition of [a, b] if they fulfill the following conditions for i = 1,2,……n .

1. $A_i : [a, b] \to [0, 1]$, $A_i(x_i) = 1$.
2. $A_i(x) = 0$ if x does not belongs to (x_{k-1}, x_{k+1})
3. $A_i(x)$ is continuous.
4. $A_i(x)$ is monotonically increasing on $[x_{k-1}, x_k]$ and monotonically decreasing on $[x_k, x_{k+1}]$.
5. $\sum_{i=n+1}^{2n} A_i(x) = 1$, for all x.
6. $A_i(x_i - x) = A_i(x_i + x)$, for all $x \in [0,h]$, $i = 2,\ldots,n-1$, $n > 2$.
7. $A_{i+1}(x) = A_i(x - h)$, for all $x \in [a+h, b]$, for $i = 2,3,\ldots,n-2$, $n > 2$.

where h is the uniform distance between two nodes.

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Definition1.2 ([6]). Let $f(x)$ be a continuous function on [a , b] and $A_1(x)$, $A_2(x)$, ……,$A_n(x)$ be basis functions determining a uniform fuzzy partition of [a , b] .The n-tuple of real numbers, [$F_1, F_2, \dots F_n$] such that

$$F_i = \frac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx}, \quad i= 1,2,\dots,n \tag{1}$$

Will be called the F- transform of f with respect to the given basis functions. Real's F_i are called components of the F-transform.

2. BIPOLAR FUZZY PARTITION OF THE UNIVERSE:

Here we take an interval [−b, b] as a universe and assume a ∈ [−b, b] be any point in the interval. Then we may partition the interval [−b, b] by fuzzy sets $A_1, A_2, \dots\dots, A_{2n}$ with membership functions $A_1(x), A_2(x), \dots\dots, A_{2n}(x)$ defined on [−b, b], if they satisfy some properties which are described below .

Definition2.1 [1]: Let [−b, b] be an interval such that $-b \le a \le b$ and $-b = x_1 < x_2 < \dots\dots < x_n = a = x_{n+1} < x_{n+2} < \dots\dots < x_{2n} = b$, be fixed nodes within [−b, b] and $n \ge 2$. We say that fuzzy sets $A_1, A_2, \dots\dots, A_{2n}$,defined on [−b, b] form a bipolar fuzzy partition if they fulfill the following conditions .

1. $A_i : [-b , b] \to [-1 ,1]$, for i = 1,2,….2n.
   $A_i(x_i) = -1$ for i = 1,2 ….n
   $A_i(x_i) = 1$ for i = n+1, n+2, …..2n.
2. $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$, for all i.
3. $A_i(x)$ is continuous, for i = 1,2,…..n □ 1,n +2,n +3, …..2n. A_n and A_{n+1} may be discontinuous at the point $x_n = a = x_{n+1}$.
4. $A_i(x)$ is monotonically decreases on [$x_{i□1}$, x_i] and increases on [x_i, x_{i+1}] , for i= 1,2…..n and reverse for i = n+1, n+2, ….2n.
5. For all $x \in [-b, a]$, $\sum_{i=1}^{n} A_i(x) = -1$ and ∀ $x \in$ [a, b] $\sum_{i=n+1}^{2n} A_i(x) = 1$.That is for x = a both the condition must be satisfied.
6. $A_i(x_i - x) = A_i(x_i + x)$, ∀ x ∈ [0,h], i = 2,…,n -1, n>2 .
   $A_i(x_i - x) = A_i(x_i + x)$, ∀x∈ [0,k], i =n +2,…,2n -1, n > 2.
7. $A_{i+1}(x) = A_i(x - h)$, ∀ x ∈ [x_i ,x_{i+1}] , for i = 2,3,….,n – 2, n > 2 . and
   $A_{i+1}(x) = A_i(x - k)$, for all x ∈ [x_i ,x_{i+1}] , for i = n+2,n+3,….,2n – 2, n > 2 .

where h and k are the uniform distance between two nodes of $x_1, x_2, \ldots, x_n$ and x_{n+1}, $x_{n+2}, \ldots \ldots x_{2n}$ respectively. The membership functions $A_1(x), A_2(x), \ldots\ldots, A_{2n}(x)$ are called basic functions.

Let us remarks that the shape of basic functions is not predetermined and, therefore it may be chosen additionally for further requirements (e.g. smoothness).

Lemma 2.1[1]: Let the uniform partition of [−b, b] be given by basic functions $A_1, A_2, \ldots\ldots, A_{2n}$ .Then

$$\int_{x_1}^{x_2} A_1(x)dx = \int_{x_{n-1}}^{x_n} A_n(x)dx = -\frac{h}{2} \tag{2}$$

For, i = 2,3,…,n-1, $\int_{x_{i-1}}^{x_{i+1}} A_i(x)dx = -h$, (3)

$$\int_{x_{n+1}}^{x_{n+2}} A_1(x)dx = \int_{x_{2n-1}}^{x_{2n}} A_n(x)dx = \frac{k}{2} \tag{4}$$

$\int_{x_{i-1}}^{x_{i+1}} A_i(x)dx = k$, for i = n+2,n+3,…,2n-1, (5)

The above lemma shows that the integral of the basic functions does not depend upon the particular shape of the basic functions.

3. BIPOLAR FUZZY TRANSFORM:

In this section we introduce a new transform called bipolar fuzzy transform (BF-transform) which establishes a relation between the set of continuous functions on [□ b ,b] and the set of 2n dimensional vectors.

Definition3.1: Let $A_1, A_2, \ldots\ldots, A_{2n}$ be basic functions which form a bipolar fuzzy partition of [− b, b] and f be any real continuous functions defined on [− b ,b] .We say that the 2n tuple of real numbers $[BF_1, BF_2, \ldots\ldots\ldots, BF_{2n}]$ given by

$$BF_i = \frac{\int_{-b}^{b} f(x)A_i(x)dx}{\int_{-b}^{b} A_i(x)dx} \tag{6}$$

$$\Rightarrow BF_i = \frac{\int_{-b}^{a} f(x)A_i(x)dx + \int_{a}^{b} f(x)A_i(x)dx}{\int_{-b}^{a} A_i(x)dx + \int_{a}^{b} A_i(x)dx}$$

is called the bipolar fuzzy transform of *f* with respect to $A_1, A_2, \ldots\ldots, A_{2n}$ .Note that our definition is correct since for each i = 1,2,…2n the product $f.A_i$ is integrable function on [−b, b] .
Now for i = n+1,n+2, ……,2n, $A_i(x) = 0$, for all x ∈ [□b, a) .

Hence $BF_i = \dfrac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx} = F_i$

where F_i is fuzzy transform components in the sense of perfilieva [6].

and for i = 1,2,….,n, $A_i(x) = 0$ for all $x \in (a, b]$.

Hence $BF_i = \dfrac{\int_{-b}^a f(x)A_i(x)dx}{\int_{-b}^a A_i(x)dx}$

Thus by using Lemma2.1, we can say that

$BF_i = F_i$, for i = n+1,n+2, …….,2n. (7)

$BF_i = -\dfrac{2}{h}\int_{-b}^a f(x)A_i(x)dx$, for i = 1 and n. (8)

$BF_i = \dfrac{1}{h}\int_{-b}^a f(x)A_i(x)dx$, for i = 2,3,…n-1. (9)

4. PROPERTIES OF *BF*- TRANSFORM:

Let *f* be any continuous function defined on [−b, b] .We denote the *BF*-transform of f with respect to $A_1, A_2, \ldots, A_{2n}$ by $BF_{2n}[f]$. The elements $BF_1, BF_2, \ldots, BF_{2n}$ are called components of the *BF*- transform.

Now from the definition of BF-transform it can be easily seen that bipolar fuzzy transform is a linear function. Since for any $f, g \in c[-b, b]$,

$$BF_{2n}[\alpha f + \beta g] = \frac{\int_{-b}^b (\alpha f + \beta g)(x)A_i(x)dx}{\int_{-b}^b A_i(x)dx}$$

$$= \alpha \frac{\int_{-b}^b f(x)A_i(x)dx}{\int_{-b}^b A_i(x)dx} + \beta \frac{\int_{-b}^b g(x)A_i(x)dx}{\int_{-b}^b A_i(x)dx}$$

$$= \alpha BF_{2n}[f] + \beta BF_{2n}[g] \text{, for any } \alpha, \beta \text{ in } R .$$

Now we investigate how well if the original *f* will be represented by its *BF*-transform. We will show that the components of the *BF*-transform are the weighted mean values of the given functions where the weights are given by the basic functions. Now we will prove the following theorem under certain conditions.

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Theorem3.1 [1]: Let f be a continuous function defined on [−b, b] and $A_1, A_2, \ldots\ldots, A_{2n}$ be basic functions which form a bipolar fuzzy partition of [−b ,b] .Then the i^{th} components of the *BF*-transform gives minimum to the function

$$\Phi(y) = \int_{-b}^{b} (f(x) - y)^2 A_i(x)dx, \text{ defined on } [f(-b), f(b)].$$

5. INVERSE BIPOLAR FUZZY TRANSFORM:

Now a question arises in the minds that can we get back the original function by its BF-transform. The answer is we can reconstruct an approximate function to the original function. For that purpose we define inverse bipolar fuzzy transform.

Definition 3.2[1]: Let $A_1, A_2, \ldots\ldots, A_{2n}$ be basic functions which form a bipolar fuzzy partition of [−b ,b] and f be a function from $c([-b, b])$ .Let $BF_{2n}[f] = [BF_1, BF_2, \ldots\ldots, BF_{2n}]$ be the bipolar fuzzy transform of *f* with respect to $A_1, A_2, \ldots\ldots, A_{2n}$. Then the function

$$f_{BF,2n}(x) = \sum_{i=1}^{2n} BF_i . A_i(x)$$

is called the inverse bipolar fuzzy transform.

Now we propose a theorem which shows that the inverse bipolar fuzzy transform can be able to approximate any continuous functions *f* with an arbitrary precision.

Theorem3.2 ([1]): Let *f* be a continuous functions defined on [−b, b]. Then for any ε > 0 there exist $2n_\varepsilon$ and a bipolar fuzzy partition $A_1, A_2, \ldots\ldots, A_{2n}$ of [−b, b] such that for all $x \in [-b, b]$

$$\left| f(x) - f_{BF,2n}(x) \right| \le \varepsilon.$$

Lemma3.1: Let *f* be twice continuously differentiable in (−b, b) .Then for each i = 1,2,…….,n

$$BF_i = f(x_i) + O(h^2) \quad (10)$$

and for i = n+1, n+2,…….,2n

$$BF_i = f(x_i) + O(h^2) \quad (11)$$

Proof: We will prove the equation (10).The proof will be given for one fixed value of i which lies between 2 and n−1. The other two cases i = 1 and n can be proof similarly. We will apply the trapezium formula with nodes x_{i-1}, x_i, x_{i+1} to the computation of the integral

$$-\frac{1}{h}\int_{x_{i-1}}^{x_{i+1}}(f(x)A_i(x)dx$$

and obtain

$$BF_i \;=\; -\frac{1}{h}\int_{x_{i-1}}^{x_{i+1}}(f(x)A_i(x)dx$$

$$= \square\, \frac{1}{h}\cdot\frac{h}{2}[\, f(x_{i-1})A_i(x_{i-1})+2f(x_i)A_i(x_i)$$

$$+f(x_{i+1})A(x_{i+1})\,] + O(h^2)$$

$$= \square\;\; f(x_i)A_i(x_i)\;\; +O(h^2)$$

since $A_i(x_{i-1})$ and $A_i(x_{i+1})$ are both zero by definition.

$$\Rightarrow BF_i \;=\; f(x_i)\;\; +O(h^2),\;\text{since}\;\; A_i(x_i) = \square 1.$$

Since for $i = n+1, n+2, \ldots\ldots, 2n$, $BF_i = F_i$, therefore equation (11) can be proof similarly by Lemma 4 in [6] .

Lemma 3.2: For any continuous function $f(x)$ the integral $\int_{-b}^{b} f(x)dx$ can be computed precisely as

$$\int_{-b}^{b} f(x)dx = h\,(\frac{1}{2}BF_1 + BF_2 + \ldots\ldots + \frac{1}{2}BF_n\,) + k(\frac{1}{2}BF_{n+1} + BF_{n+2} \;+ \ldots\ldots + \;\frac{1}{2}\; BF_{2n}\,)\,.$$

The following theorem estimates the difference between the approximation of a given continuous functions by inverse bipolar fuzzy transform based on two different sets of basic functions.

Theorem3.3 [1] : Let f be any continuous function defined on [−b, b] and A'_1 , $A'_2,\ldots\ldots,A'_{2n}$ as well as $A''_1, A''_2,\ldots., A''_{2n}$, $n \geq 3$ be two sets of basic functions which form different uniform bipolar fuzzy partitions of [−b, b] . Let $f'_{BF,2n}$ and $f''_{BF,2n}$ be the two inverse bipolar fuzzy transform of f with respect to different sets of basic functions. Then

$$|\, f'_{BF,2n}(x) - f''_{BF,2n}(x)\, | \;\leq 2\omega\,(2h, f\,),\ \text{for}\ x \in\ [-b\ ,a]$$

and $$|\, f'_{BF,2n}(x) \;\square\; f''_{BF,2n}(x)\, | \;\leq 2\omega\,(2k, f\,),\ \text{for}\ x \in\ [a\ ,b]\ .$$

where 2ω(2h,f) and 2ω(2k,f) are the modulus of continuity of f on the interval [−b, a] and [a ,b] respectively .

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Now we will discuss an important property of bipolar fuzzy transform. In this section, we will show that monotonocity of a function remains invariant under bipolar fuzzy transform.

6. MONOTONOCITY

Let M [−b, b] be a class of bounded monotone functions on [−b, b]. Then, these functions are integrable and thus their bipolar fuzzy transform exists. For certainty, we will assume that, M [−b, b] consists of monotonically increasing functions. Let $A_1, A_2, \ldots\ldots, A_{2n}$ be a bipolar fuzzy partition of [−b, b]. Our aim is to show that for any function $f \in M$ [−b, b], both the bipolar fuzzy transform of *f and* its inverse are monotonically increasing functions on the respective domains. The BF-transform $BF_{2n}[f] = [BF_1, BF_2, \ldots\ldots\ldots, BF_{2n}]$ is a function on the domain { 1,2,…….., 2n} , so that for $k = 1,2,\ldots\ldots\ldots,2n$, $BF_{2n}[f](k) = BF_k$.

The following technique will simplify the proof presented below. It consists of extending [−b, b] to [−b −h, b + k] and extending each functions f from M [−b, b] and basic functions $A_1, A_n, A_{n+1}, A_{2n}$ to the following ones

$$f^e(x) = \begin{cases} f(-b-x) = 2f(-b) - f(-b+x), x \in [0,h] \\ f(x), x \in [-b,b] \\ f(b+x) = 2f(b) - f(b-x), x \in [0,k] \end{cases}$$

$$A_1^e(x) = \begin{cases} A_1(-b-x) = A_1(-b+x), x \in [0,h] \\ A_1(x), x \in [-b,-b+h] \end{cases}$$

$$A_n^e(x) = \begin{cases} A_n(x), x \in [a-h,a] \\ A_n(a+x) = A_n(a-x), x \in [-0,h] \end{cases}$$

$$A_{n+1}^e(x) = \begin{cases} A_{n+1}(a-x) = A_{n+1}(a+x), x \in [0,k] \\ A_{n+1}(x), x \in [a,a+k] \end{cases}$$

$$A_{2n}^e(x) = \begin{cases} A_{2n}(x), x \in [b-k,b] \\ A_{2n}(b+x) = A_{2n}(b-x), x \in [0,k] \end{cases}$$

Moreover, we denote $x_0 = -$ b − h, $x_{2n+1} = b + k$.

Now let us replace the bipolar fuzzy transform components BF_1 , BF_n , BF_{n+1} , BF_{2n} of f by

$$BF_1^e = \frac{1}{-h}\int_{-b-h}^{-b+h} f^e(x) A_1^e(x)dx$$

$$BF_n^e = \frac{1}{-h}\int_{a-h}^{a+h} f^e(x)A_n^e(x)dx$$

$$BF_{n+1}^e = \frac{1}{k}\int_{a-k}^{a+k} f^e(x)A_{n+1}^e(x)dx$$

$$BF_{2n}^e = \frac{1}{k}\int_{b-k}^{b+k} f^e(x)A_{2n}^e(x)dx$$

So that the inverse bipolar fuzzy transform of f will change to

$$f_{BF,2n}(x) = BF_1^e.A_1^e(x) + \sum_{i=2}^{n-1} BF_i.A_i(x) + BF_n^e.A_n^e(x),$$

for all $x \in [-b, a]$, and

$$f_{BF,2n}(x) = BF_{n+1}^e.A_{n+1}^e(x) + \sum_{i=n+2}^{2n-1} BF_i.A_i(x) + BF_{2n}^e.A_{2n}^e(x)$$

for all $x \in [a, b]$.

That is both inverse bipolar fuzzy transform of f and f^e, coincide on $[x_2, x_{n-1}] \cup [x_{n+1}, x_{2n-1}]$.

In the following section we will consider the extended interval [−b −h, b +k] as a universal set and will write the extended function, extended bipolar fuzzy partition and the extended bipolar fuzzy transform components as in the original notation and without using the notation ^{e}.

Lemma4.1: Let $f \in M$ [−b, b] and $BF_{2n}[f] = [BF_1, BF_2, \ldots\ldots\ldots, BF_{2n}]$ be the bipolar fuzzy transform of the extended f with respect to the extended bipolar fuzzy partition $A_1, A_2, \ldots\ldots, A_{2n}$. Then $BF_{2n}[f]$ is a monotonically increasing function on the domain { 1,2,………,2n} ,such that

$$BF_1 \le BF_2 \le \ldots\ldots\ldots, \le BF_{2n}.$$

Proof: The proof of the lemma will be given in three cases.
Case1. For k = 1,2, …….., n−1.
Case2. For k = n+2, n+3, ……., 2n−1

Case3. For k = n, n+1 .

We will discuss only case1; remaining cases can be done similarly.

Let k be on 1,2,………,n −1 ,then

$$-h(BF_{k+1} - BF_k) = \int_{x_k}^{x_{k+2}} f(x)A_{k+1}(x)dx - \int_{x_k}^{x_{k+2}} f(y-h)A_{k+1}(y)dy$$

$$= \int_{x_k}^{x_{k+2}} (f(x) - f(x-h))A_{k+1}(x)dx \leq 0$$

Since, f is monotonically increasing and therefore,

$f(x) - f(x-h) \geq 0$ and $A_{k+1}(x) \leq 0.$

Hence, BF_{k+1} □ $BF_k \geq 0$

$\Rightarrow BF_{k+1} \geq BF_k$

The remaining cases can also be proved similarly.

Theorem 4.2: Let $f \in M$ [−b, b] and $BF_{2n}[f] = [BF_1, BF_2, \ldots\ldots, BF_{2n}]$ be the bipolar fuzzy transform of the extended f with respect to the extended bipolar fuzzy partition $A_1, A_2, \ldots\ldots, A_{2n}$. Then the inverse bipolar fuzzy transform of the extended f is a monotonically increasing function on the interval [−b, b].

Proof: Let x , $y \in$ [−b, b], be such that $x \leq y$. We will prove that $f_{BF,2n}(x) \leq f_{BF,2n}(y)$. This proof is based on lemma 4.1 and is split on several cases according to the position of x,y with respect to the nodes x_1, x_2, x_3 ……,x_{2n} of the extended partition.

Case1: Assume that for some k = 1, 2..…, n−1, and $x, y \in [x_k, x_{k+1}]$. Then

$$f_{BF,2n}(y) - f_{BF,2n}(x) = \sum_{i=1}^{n} BF_i.A_i(y) - \sum_{i=1}^{n} BF_i.A_i(x)$$

$$= BF_k(A_k(y) - A_k(x)) + BF_{k+1}(A_{k+1}(y) - A_{k+1}(x))$$

$$= BF_k(A_k(y) - A_k(x)) + BF_{k+1}(-1 - A_k(y) + 1 + A_k(x))$$

$$= (A_k(x) - A_k(y))(BF_{k+1} - BF_k) \geq 0$$

since by lemma4.1, $(BF_{k+1} - BF_k) \geq 0$,and for this case $(A_k(x) - A_k(y)) \geq 0$, as A_k is monotonically increases on

$[x_k, x_{k+1}]$ and A_k is negative.

Case2: Assume that for some k = n+1,n+ 2..…, 2n−1, and $x, y \in [x_k, x_{k+1}]$. Then

$$f_{BF,2n}(y) - f_{BF,2n}(x) = \sum_{i=1}^{n} BF_i.A_i(y) - \sum_{i=1}^{n} BF_i.A_i(x)$$

$$= BF_k(A_k(y) - A_k(x)) + BF_{k+1}(A_{k+1}(y) - A_{k+1}(x))$$

$$= BF_k(A_k(y) - A_k(x)) + BF_{k+1}(1 - A_k(y) - 1 + A_k(x))$$

$$= (A_k(x) - A_k(y))(BF_{k+1} - BF_k) \geq 0$$

since by lemma4.1, $(BF_{k+1} - BF_k) \geq 0$, and for this case $(A_k(x) - A_k(y)) \geq 0$, as A_k is monotonically decreases on [x_k, x_{k+1}] and A_k is positive.

Similarly, all the remaining cases can be proof easily. So we omit the proof here.

7. CONCLUSION

In this paper we have discussed about a new technique called bipolar fuzzy transform and some of its properties. This new transform (fuzzy bipolar transform) is a generalization of the fuzzy transform developed by Irina Perfilieva. This paper also investigates the monotonicity of functions that are invariant with respect to bipolar fuzzy transform. We proved that the bipolar fuzzy transform of any extended monotone function f is a monotone function, as is the inverse bipolar fuzzy transform of f.

We also established that the approximation of a continuous function by inverse bipolar fuzzy transform converges uniformly to the original functions.

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