FUZZINESS ON HYPERBOLIC KAC-MOODY ALGEBRAS HG_2

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Abstract: Kac-Moody algebras is one of the modern field of Mathematical research which has got interesting connections and applications to other fields of Mathematics and Mathematical Physics. On the other hand the theory of fuzzy sets and logic is applied to various fields of Mathematics and other branches of science. In this paper, fuzzy set is defined on the Cartesian product of root basis of two hyperbolic type of Kac-Moody algebras, which are extensions of G_2. Basic fuzzy properties on these algebras are studied; For specific values of α, α - cuts and strong α - cuts are computed. For the fuzzy set defined on the Cartesian product of root basis of G_2, the convexity is verified;

Keywords: α - level sets, Fuzzy set, Hyperbolic type, Kac-Moody algebra

1. INTRODUCTION

1.1 Basic definitions on Kac-Moody algebras

Definition 1[3]: An integer matrix $A = (a_{ij})_{i,j=1}^{n}$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

i) $a_{ii} = 2 \ \forall$ i =1,2,….,n
ii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \ \forall$ i, j = 1,2,…,n
iii) $a_{ij} \leq 0 \ \forall$ i, j = 1,2,…,n.

Let us denote the index set of A by N = {1,…, n}. A GCM A is said to decomposable if there exist two non-empty subsets I, J $\subset$ N such that I $\cup$ J = N and $a_{ij} = a_{ji} = 0 \ \forall$ $i \in I$ and $j \in J$. If A is not decomposable, it is said to be indecomposable.

Definition 2[2]: A realization of a matrix $A = (a_{ij})_{i,j=1}^{n}$ is a triple (H, Π, Π^{v}) where l is the rank of A, H is a 2n - l dimensional complex vector space, $\Pi = \{\alpha_1,..., \alpha_n\}$ and $\Pi^{v} = \{\alpha_1^{v},..., \alpha_n^{v}\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^{v}) = a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are called simple roots. The root lattice generated by Π is $Q = \sum_{i=1}^{n} Z\alpha_i$.

Definition 3[2] :The Kac-Moody algebra g(A) associated with a GCM $A = (a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i, f_i, $i = 1,2,...,n$ and H with the following defining relations :

$[h, h^{'}] = 0, h, h^{'} \in H$, $[e_i, f_j] = \delta_{ij}\alpha_i^v$

$[h, e_j] = \alpha_j(h)e_j$, $[h, f_j] = -\alpha_j(h)f_j$, $i, j \in N$ The Kac-Moody

$(ad \ e_i)^{1-a_{ij}} e_j = 0, (ad \ f_i)^{1-a_{ij}} f_j = 0 \ \forall$ i ≠ j, i, j ∈ N

algebra g(A) has the root space decomposition $g(A) = \underset{\alpha \in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A) = \{x \in$

$g(A) / [h, x] = \alpha(h)x \ for \ all \ h \in H\}$. An element α, $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$. Let $Q_+ = \sum_{i=1}^{n} Z_+\alpha_i$. Q has a partial ordering " ≤" defined by $\alpha \leq \beta$ if $\beta - \alpha \in Q_+$, where $\alpha, \beta \in Q$.

Definition 4[2]: Let $\Delta (= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Definition 5[2]: To every GCM A is associated a Dynkin diagram S (A) defined as follows: S(A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|, |a_{ji}|\}$ number of lines if $a_{ij} . a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $a_{ij}. a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|, |a_{ji}|)$ of integers.

Theorem 6[2]: Let A be a real n x n matrix satisfying (m1), (m2) and (m3).

(m1) A is indecomposable;

(m2) $a_{ij} \leq 0$ for $i \neq j$;

(m3) $a_{ij} = 0$ implies $a_{ji} = 0$

Then one and only one of the following three possibilities holds for both A and ^tA:

(1) det A ≠ 0; there exists u > 0 such that A u > 0; Av ≥ 0 implies v > 0 or v = 0;

(2) co rank A=1; there exists u > 0 such that Au = 0; Av ≥ 0 implies Av = 0;

(3) there exists u > 0 such that Au < 0; Av ≥ 0, v ≥ 0 imply v = 0.

Then A is of finite, affine or indefinite type iff (1), (2) or (3) is satisfied.

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Definition 7[5]: A Kac- Moody algebra g(A) is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.

Definition 8[5]: An indecomposable GCM A is said to be of hyperbolic type if it is of indefinite type and any connected proper sub diagram of S(A) is of finite or affine type.

1.2 Basic definitions on fuzzy sets

Definition 9[6]: A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$ that can be finite, countable or over countable.

Definition 10[6]: If X is a collection of objects denoted generically by *x*, then a fuzzy set $\tilde{A}$ is defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / x \in X\}$. $\mu_{\tilde{A}}(x)$ is called the membership function or "grade of membership" of x in $\tilde{A}$ that maps X to the membership space M.

Definition 11[6]: The support of a fuzzy set $\tilde{A}$, $S(\tilde{A})$ is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

Definition 12[6]: The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α - level set $A_\alpha = \{x \in X / \mu_{\tilde{A}}(x) \geq \alpha\}$; $A_\alpha^{'} = \{x \in X / \mu_{\tilde{A}}(x) > \alpha\}$ is called "Strong α - level set" or "Strong α - cut".

Definition 13[6]: Let $\tilde{A}$ be a fuzzy set on X. Then the set $\{x \in X / \mu_{\tilde{A}}(x) = 1\}$ is called the core of the fuzzy set $\tilde{A}$. This set is denoted by core ($\tilde{A}$).

Definition 14[6]: A fuzzy set $\tilde{A}$ is said to be normal if $\sup_x \mu_{\tilde{A}}(x) = 1$.

Definition 15[6]: The membership function of the complement of a normalized fuzzy set $\tilde{A}$, $\mu_{\complement\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) \quad x \in X$.

Definition 16[6]: For a finite fuzzy set $\tilde{A}$, the cardinality $|\tilde{A}|$ is defined as $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)$. $\|\tilde{A}\| = |\tilde{A}| / |X|$ is called the relative cardinality of $\tilde{A}$.

Definition 17[6]: A fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1+(1-\lambda)x_2)\geq \min\{\mu_{\tilde{A}}(x_1),\mu_{\tilde{A}}(x_2)\},\ x_1,x_2\in X,\ \lambda\in[0,1].$

Definition 18[1]: Let A be a fuzzy set on U and α be a number such that 0 < α $\leq$ 1. Then by αA we mean a fuzzy set on U, denoted by αA which is such that $(\alpha A)(x) = \alpha A(x)$ for every x in U. This procedure of associating another fuzzy set with the given fuzzy set A is termed as restricted scalar multiplication.

Theorem 19. [1] Any fuzzy set A on U can be decomposed as A = sup $\{\alpha A_\alpha \,/\, 0 < \alpha \leq 1\}$. We also write $A=\sum \alpha A_\alpha$ or $A = \cup \alpha A_\alpha$.

As in [4], the new concept of fuzzy sets on the root basis of Kac- Moody algebras was introduced. Let g(A) be the Kac-Moody algebra with the GCM A. Let Π be the root basis of g(A). Then the fuzzy set on $X=\Pi\times\Pi$ is defined as follows:

$$\mu_{\tilde{A}}(\alpha_i,\alpha_j)=\begin{cases}1/\max(|\,a_{ij}\,|,|\,a_{ji}\,|) & if \quad a_{ij}\neq 0\\ 0 & if \quad a_{ij}=0\end{cases}$$

$$for \quad (\alpha_i,\alpha_j)\in X,\ i,\ j=1,2,...,\ l \qquad (1)$$

Then $\tilde{A}=((\alpha_i,\alpha_j),\mu_{\tilde{A}}(\alpha_i,\alpha_j))$ forms a fuzzy set on $\Pi\times\Pi$

The following basic properties of fuzzy set defined by equation (1) are also given in [4].

(i) Support of $\tilde{A}$ consists of all (α_i , α_j) such that $a_{ij} \neq 0$, for i, j = 1,…, *l*.

(ii) Core of the fuzzy set $\tilde{A}$ is non – empty if and only if the associated Dynkin diagram contains at least one single edge ○—○ .

(iii) The fuzzy set $\tilde{A}$ defined by (1) is normal iff a single edge ○—○ occurs in the Dynkin diagram associated with the GCM A.

2. SOME FUZZY PROPERTIES ON THE ROOT BASIS OF KAC-MOODY ALGEBRA:

Lemma 20: A fuzzy set $\tilde{A}$ corresponding to the finite Kac-Moody algebra G_2 defined by equation (1) is convex.

Proof: Consider the finite type of Kac-Moody algebra G_2. The following table 1 shows all possible membership grades for the elements of X and the condition for checking convexity is verified:

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Table 1. Possible membership grades attained by the elements in $\Pi \times \Pi$

$\mu_{\tilde{A}}(\alpha_i,\alpha_j)$	$\mu_{\tilde{A}}(\alpha_k,\alpha_l)$	$\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l))$	$\min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j),\mu_{\tilde{A}}(\alpha_k,\alpha_l)\}$
1/2	1/2	1/2	1/2
1/2	1/3	$(\lambda+2)/6$	1/3
1/3	1/2	$(3-\lambda)/6$	1/3
1/3	1/3	1/3	1/3

For every element in $\Pi \times \Pi$, we see that the following inequality is satisfied:

$$\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l)) \geq \min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j),\mu_{\tilde{A}}(\alpha_k,\alpha_l)\},\ \lambda \in [0,1].$$

Hence the fuzzy set $\tilde{A}$ corresponding to the hyperbolic type of Kac-Moody algebra G_2 is convex.

Next, consider the hyperbolic Kac-Moody algebra HG_2, associated with the GCM A $= \begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

whose Dynkin diagram is given by

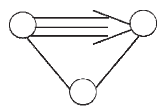

Fig.1. Dynkin diagram for the hyperbolic type of Kac- Moody algebra

Let $\tilde{A}$ be the fuzzy set defined on A given by the equation (1). Then the following lemmas give the basic properties of fuzzy sets defined on X.

Lemma 21: Support of $\tilde{A} = \{(\alpha_1,\alpha_3),(\alpha_2,\alpha_3),(\alpha_3,\alpha_1),(\alpha_3,\alpha_2),$

$(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1)\}$

Proof: By the definition of support,

Supp($\tilde{A}$) = $\{(\alpha_i,\alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i,\alpha_j) > 0\}$.

From the GCM, it is clear that the Supp($\tilde{A}$) $= \{(\alpha_1,\alpha_3),$ $(\alpha_2,\alpha_3),(\alpha_3,\alpha_1),(\alpha_3,\alpha_2),(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1)\}$ Lemma 22: Height of the fuzzy set $\tilde{A}$ is 1.

Proof: From the GCM entries of A, we can see that the maximum membership grade attained is 1. Hence the height of the fuzzy set $\widetilde{A}$ is 1.

Lemma 23: Core($\widetilde{A}$) $=\{(\alpha_1,\alpha_3),(\alpha_2,\alpha_3),(\alpha_3,\alpha_1),(\alpha_3,\alpha_2)\}$

Proof: By the definition of core,

Core($\widetilde{A}$)= $\{(\alpha_i,\alpha_j)/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)=1\}$

$$=\{(\alpha_1,\alpha_3),(\alpha_2,\alpha_3),(\alpha_3,\alpha_1),(\alpha_3,\alpha_2)\}$$

Lemma 24: The fuzzy set $\widetilde{A}$ is normal.

Proof: From the membership grades of elements of X, we get $Sup_{(\alpha_i,\alpha_j)}\mu_{\widetilde{A}}(\alpha_i,\alpha_j)=1$. Hence the fuzzy set is normal.

Lemma 25: Let $\widetilde{A}$ be the fuzzy set defined on $\Pi\times\Pi$ for the hyperbolic type of Kac-Moody algebra associated with the GCM A= $\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ given by equation (1). Then $\widetilde{A}$ has the following properties:

(a) The cardinality | $\widetilde{A}$ |= 6.17

(b) Relative cardinality || $\widetilde{A}$ || = 0.69

(c)The membership function of the complement of a normalized fuzzy set $\widetilde{A}$ corresponding to this hyperbolic type of Kac-Moody algebra is listed below:

For $i=2$ $\mu_{\complement\widetilde{A}}(\alpha_{i-1},\alpha_i)=$ $\mu_{\complement\widetilde{A}}(\alpha_i,\alpha_{i-1})=2/3$;

For $i=3$ $\mu_{\complement\widetilde{A}}(\alpha_{i-1},\alpha_i)=$ $\mu_{\complement\widetilde{A}}(\alpha_i,\alpha_{i-1})=0$;

For $i=3$ $\mu_{\complement\widetilde{A}}(\alpha_i,\alpha_{i-2})=$ $\mu_{\complement\widetilde{A}}(\alpha_{i-2},\alpha_i)=0$;

For $i=1,2,3$ $\mu_{\complement\widetilde{A}}(\alpha_i,\alpha_i)=1/2$.

Proof: (a) The fuzzy set $\widetilde{A}$ corresponding to the hyperbolic type of Kac-Moody algebra associated with the GCM A= $\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ contains 4 elements in X having membership grade 1, 3 elements in X having membership grade 1/2 and 2

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elements in X having membership grade 1/3. By the definition of cardinality, $|\tilde{A}| = \sum_{x\in X} \mu_{\tilde{A}}(x) = 6.17$

(b) $\|\tilde{A}\| = |\tilde{A}| / |X| = 0.69$.

(c) The fuzzy set $\tilde{A}$ corresponding to the hyperbolic type of Kac-Moody algebra associated with the GCM A= $\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, is normal. For $(\alpha_i, \alpha_j) \in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ is given below:

For $i=2$ $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1/3 = 2/3$;

For $i=3$ $\mu_{\complement\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1 = 0$;

For $i=3$ $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_{i-2}) = \mu_{\complement\tilde{A}}(\alpha_{i-2}, \alpha_i) = 1 - 1 = 0$;

For $i = 1,2,3$ $\mu_{\complement\tilde{A}}(\alpha_i, \alpha_i) = 1 - 1/2 = 1/2$.

Computation of α – level sets: We shall now determine the α – level sets and strong α – level sets for some specific cases of GCM A, for this hyperbolic type of Kac-Moody algebras.

Theorem 26: For the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A = $\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ given by the equation (1). Then the α - level sets and strong α - level sets for $\alpha = 1, 1/2, 1/3, \ldots, 1/k, \ldots$ are given below:

$(i) A_1 = \{(\alpha_1, \alpha_3), (\alpha_2, \alpha_3), (\alpha_3, \alpha_1), (\alpha_3, \alpha_2)\}$

$(ii) A_{1/2} = A_1 \cup \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_3, \alpha_3)\}$

$(iii) A_{1/3} = A_{1/2} \cup \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\} = X$

$(iv) A_{1/2}' = A_1$

$(v) A_{1/3}' = A_{1/2}$

$(vi) A_{1/4}' = A_{1/3}$

(vii) $|A_1| = 4$, $|A_{1/2}| = 7$, $|A_{1/3}| = 9$, $|A_{1/4}'| = 9$. *For* $k = 4,5,\ldots,$ $|A_{1/k}| = |A_{1/k}'| = 9$.

Proof: Consider the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A = $\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$.

$$(i) A_1 = \{(\alpha_i,\alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i,\alpha_j) \geq 1\}$$
$$= \{(\alpha_i,\alpha_j) \in X / \quad 1/\max(|a_{ij}|,|a_{ji}|) \geq 1\}$$
$$= \{(\alpha_1,\alpha_3),(\alpha_2,\alpha_3),(\alpha_3,\alpha_1),(\alpha_3,\alpha_2)\}$$

$$(ii) A_{1/2} = \{(\alpha_i,\alpha_j) \in X / \quad 1/\max(|a_{ij}|,|a_{ji}|) \geq 1/2\}$$
$$= A_1 \cup \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3)\}$$

$$(iii) A_{1/3} = \{(\alpha_i,\alpha_j) \in X / \quad 1/\max(|a_{ij}|,|a_{ji}|) \geq 1/3\}$$
$$= A_{1/2} \cup \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1)\} = X$$

From the above relation we have,

$$A_1 \subset A_{1/2} \subset A_{1/3} = \ldots = A_{1/k} = \ldots$$
$$(iv) A_{1/2}' = \{(\alpha_i,\alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i,\alpha_j) > 1/2\}$$
$$= \{(\alpha_i,\alpha_j) \in X / \quad 1/\max(|a_{ij}|,|a_{ji}|) > 1/2\} = A_1 = \Phi$$
$$(v) A_{1/3}' = \{(\alpha_i,\alpha_j) \in X / \quad 1/\max(|a_{ij}|,|a_{ji}|) > 1/3\} = A_{1/2}$$
$$(vi) A_{1/4}' = \{(\alpha_i,\alpha_j) \in X / \quad 1/\max(|a_{ij}|,|a_{ji}|) > 1/4\} = A_{1/3}$$

From the above relations, we see that,

$$A_1' \subset A_{1/2}' \subset A_{1/3}' \subset A_{1/4}' = \ldots = A_{1/k}' = \ldots$$

(v) By the above relations,

$$A_{1/3} = A_{1/4} = \ldots = A_{1/k} = \ldots = A_{1/4}' = \ldots = A_{1/k}' = \ldots$$

$|A_1| = 4,\ |A_{1/2}| = 7,\ |A_{1/3}| = 9,\ |A_{1/4}'| = 9.$ *For* $k = 4,5,\ldots,$
$|A_{1/k}| = |A_{1/k}'| = 9.$

Lemma 27: A fuzzy set $\tilde{A}$ corresponding to the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A = $\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, defined by equation (1) is convex.

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Proof: Consider the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A $=\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. The table 2 showing all possible membership grades for the elements of X and the conditions for checking convexity is listed below:

Table 2. Possible membership grades attained by the elements in $\Pi \times \Pi$

$\mu_{\tilde{A}}(\alpha_i,\alpha_j)$	$\mu_{\tilde{A}}(\alpha_k,\alpha_l)$	$\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l))$	$\min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j),\mu_{\tilde{A}}(\alpha_k,\alpha_l)\}$
1	1	1	1
1	1/3	$(2\lambda+1)/3$	1/3
1	1/2	$(\lambda+1)/2$	1/2
1/3	1	$(3-2\lambda)/3$	1/3
1/3	1/3	1/3	1/3
1/3	1/2	$(3-\lambda)/6$	1/3
1/2	1	$(2-\lambda)/2$	1/2
1/2	1/3	$(\lambda+2)/6$	1/3
1/2	1/2	1/2	1/2

For every element in $\Pi \times \Pi$, we see that the following inequality is satisfied:

$$\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l)) \geq \min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j),\mu_{\tilde{A}}(\alpha_k,\alpha_l)\},\ \lambda \in [0,1].$$

Hence the fuzzy sets $\tilde{A}$ corresponding to the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A is convex.

Lemma 28: Let $\tilde{A}$ be the fuzzy set on $\Pi \times \Pi$, where Π denotes the root basis for the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A $=\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ given by equation (1). Then the α - cut decomposition for the fuzzy set $\tilde{A}$ on the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A is $\tilde{A} = 1\ A_1 \cup 1/2\ A_{1/2} \cup 1/3\ A_{1/3}$.

Proof: From Theorem 25, for the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A , $A_1 \subset A_{1/2} \subset A_{1/3}$ and $A_{1/3} = X$.

By definition, the α – cut decomposition for the fuzzy set $\tilde{A}$ is $\cup\ \alpha A_\alpha$. Hence $\tilde{A} = 1\ A_1 \cup 1/2\ A_{1/2} \cup 1/3\ A_{1/3}$.

Next, consider the hyperbolic Kac-Moody algebra belonging to the family HG_2, associated with the GCM A = $\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$ whose Dynkin diagram is given by

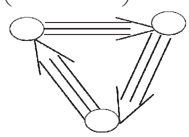

Fig.2. Dynkin diagram for the hyperbolic type of Kac- Moody algebra

Let $\tilde{A}$ be the fuzzy set defined on A given by the equation (1). Then the following lemmas give the basic properties of fuzzy sets defined on X.

Lemma 29: Supp $\tilde{A} = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_1,\alpha_2),$

$(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_3,\alpha_1),(\alpha_1,\alpha_3)\}$

Proof: By the definition of support,

Supp($\tilde{A}$)= $\{(\alpha_i,\alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i,\alpha_j) > 0\}$.

From the GCM, it is clear that the

Supp $(\tilde{A}) = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_1,\alpha_2),$

$(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_3,\alpha_1),(\alpha_1,\alpha_3)\}$

Lemma 30: Height of the fuzzy set $\tilde{A}$ is 1/2.

Proof: From the GCM entries of A, we can see that the maximum membership grade attained is 1/2. Hence the height of the fuzzy set $\tilde{A}$ is 1/2.

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Lemma 31: Core($\tilde{A}$) $= \Phi$

Proof: By the definition of core,

Core($\tilde{A}$) = $\{(\alpha_i, \alpha_j) / \mu_{\tilde{A}}(\alpha_i, \alpha_j) = 1\} = \Phi$

Lemma 32: The fuzzy set $\tilde{A}$ is not normal.

Proof: From the membership grades of elements of X, we get $Sup_{(\alpha_i, \alpha_j)} \mu_{\tilde{A}}(\alpha_i, \alpha_j) = 1/2$. Hence the fuzzy set is not normal.

Lemma 33: Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the hyperbolic type of Kac-Moody algebra associated with the GCM A = $\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$ given by equation (1) then $\tilde{A}$ has the following properties:

(a) The cardinality | $\tilde{A}$ | = 3.5

(b) Relative cardinality || $\tilde{A}$ || = 0.39

Proof: The fuzzy set $\tilde{A}$ corresponding to the hyperbolic type of Kac-Moody algebra associated with the GCM A = $\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$ contains 3 elements in X having membership grade 1/2 and 6 elements in X having membership grade 1/3. By the definition of cardinality, $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = 3.5$.

(b) $\|\tilde{A}\| = |\tilde{A}| / |X| = 0.39$.

Theorem 34: For the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A= $\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$, let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ given by the equation (1). Then the α - level sets and strong α - level sets for $\alpha = 1$, 1/2 , 1/3 ,..., $1/k$,.... are given below:

$(i) A_1 \quad = \Phi$

$(ii) A_{1/2} \quad = \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_3, \alpha_3)\}$

$(iii) A_{1/3} = A_{1/2} \cup \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_2,\alpha_3),$
$(\alpha_3,\alpha_1),(\alpha_1,\alpha_3)\} = X$

$(iv) A_{1/2}{}' = A_1$

$(v) A_{1/3}{}' = A_{1/2}$

$(vi) A_{1/4}{}' = A_{1/3}$

(vii) $|A_{1/2}| = 3$, $|A_{1/3}| = 9$, $|A_{1/4}{}'| = 9$. *For* $k = 4,5,...,$
$|A_{1/k}| = |A_{1/k}{}'| = 9.$

Proof: Consider the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A = $\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$.

$$(i) A_1 = \{(\alpha_i,\alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i,\alpha_j) \geq 1\}$$

$$= \{(\alpha_i,\alpha_j) \in X / \ 1/\max(|a_{ij}|,|a_{ji}|) \geq 1\} = \Phi$$

$$(ii) A_{1/2} = \{(\alpha_i,\alpha_j) \in X / \ 1/\max(|a_{ij}|,|a_{ji}|) \geq 1/2\}$$

$$= \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3)\}$$

$$(iii) A_{1/3} = \{(\alpha_i,\alpha_j) \in X / \ 1/\max(|a_{ij}|,|a_{ji}|) \geq 1/3\}$$

$= A_{1/2} \cup \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_2,\alpha_3),(\alpha_3,\alpha_1),(\alpha_1,\alpha_3)\} = X$ From the above relation we have,

$$A_1 \subset A_{1/2} \subset A_{1/3} = ... = A_{1/k} = ...$$

$$(iv) A_{1/2}{}' = \{(\alpha_i,\alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i,\alpha_j) > 1/2\}$$

$$= \{(\alpha_i,\alpha_j) \in X / \ 1/\max(|a_{ij}|,|a_{ji}|) > 1/2\} = A_1 = \Phi$$

$$(v) A_{1/3}{}' = \{(\alpha_i,\alpha_j) \in X / \ 1/\max(|a_{ij}|,|a_{ji}|) > 1/3\} = A_{1/2}$$

$$(vi) A_{1/4}{}' = \{(\alpha_i,\alpha_j) \in X / \ 1/\max(|a_{ij}|,|a_{ji}|) > 1/4\} = A_{1/3}$$

From the above relations, we see that,

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$A_1' \subset A_{1/2}' \subset A_{1/3}' \subset A_{1/4}' = ... = A_{1/k}' = ...$

(v) By the above relations,

$A_{1/3} = A_{1/4} = ... = A_{1/k} = ... = A_{1/4}' = ... = A_{1/k}' = ...$

$| A_{1/2} |= 3, | A_{1/3} |= 9, | A_{1/4}' |= 9.$ *For* $k = 4,5,...,$
$| A_{1/k} | = | A_{1/k}' | = 9.$

Lemma 35: A fuzzy set $\tilde{A}$ corresponding to the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A = $\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$, defined by (1) is convex.

Proof: Consider the hyperbolic type of Kac-Moody algebra associated with the above indecomposable GCM A. The table showing all possible membership grades for the elements of X and the conditions for checking convexity is listed below:

Table 3. Possible membership grades attained by the elements in $\Pi \times \Pi$

$\mu_{\tilde{A}}(\alpha_i,\alpha_j)$	$\mu_{\tilde{A}}(\alpha_k,\alpha_l)$	$\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l))$	$\min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j),\mu_{\tilde{A}}(\alpha_k,\alpha_l)\}$
1/2	*1/2*	*1/2*	*1/2*
1/2	*1/3*	$(\lambda+2)/6$	*1/3*
1/3	*1/2*	$(3-\lambda)/6$	*1/3*
1/3	*1/3*	*1/3*	*1/3*

For every element in $\Pi \times \Pi$, we see that the following inequality is satisfied:

$$\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l)) \ge \min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j),\mu_{\tilde{A}}(\alpha_k,\alpha_l)\},\ \lambda \in [0,1].$$

Hence the fuzzy sets $\tilde{A}$ corresponding to the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A is convex.

Lemma 36: Let $\tilde{A}$ be the fuzzy set on $\Pi \times \Pi$, where Π denotes the root basis for the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A = $\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$ given by equation (1). Then the α - cut decomposition for the fuzzy set $\tilde{A}$ on the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A is $\tilde{A} = 1/2\ A_{1/2} \cup 1/3\ A_{1/3}$.

Proof: From Theorem 33, for the hyperbolic type of Kac-Moody algebra associated with the indecomposable GCM A= $\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$, $A_1 \subset A_{1/2} \subset A_{1/3}$ and $A_{1/3} = X$. By definition, the α – cut decomposition for the fuzzy set $\tilde{A}$ is $\cup\ \alpha A_\alpha$. Hence $\tilde{A} = 1/2\ A_{1/2} \cup 1/3\ A_{1/3}$.

3. CONCLUSION

We can further compute the level sets for various families of affine, indefinite, hyperbolic, extended hyperbolic and non hyperbolic type of Kac-Moody algebras; Other interesting structural properties on the fuzzy nature of these algebras can also be studied.

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