EXISTENCE AND UNIQUENESS OF FUZZY SOLUTION FOR THE NONLINEAR IMPULSIVE FUZZY NEUTRAL INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITION

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Abstract: In this paper, the existence and uniqueness of a fuzzy solution for the impulsive nonlinear fuzzy integrodifferential equation with nonlocal condition is established via the Banach fixed-point theorem approach and using the fuzzy number whose values are normal, convex, upper semicontinuous, and compactly supported interval.

Keywords: Fuzzy set, fuzzy number, impulsive fuzzy integrodifferential system,non- local conditions, fuzzy solution, fixed point theorem.

1.INTRODUCTION

In various fields of engineering and physics, many problems that are related to linear viscoelasticity, nonlinear elasticity have mathematical models and are described by the problems of differential or integral equations or integrodifferential equations. Integrodifferential equations are encountered in many areas of science and technology. It is well-known that the notion of aftereffect introduced in physics is very important. To model processes with aftereffect or delay it is not sufficient to employ ordinary or partial differential equations. An approach to resolve this problem is to use integrodifferential equations. Especially, one always describes a model which possesses hereditary properties by integrodifferential equations in practice. The word "fuzzy" means "vagueness". Fuzziness occurs when the boundary of a piece of information is not clear-cut. Fuzzy set have been introduced by Zadeh [21] as an extension of the classical notion of set. Classical set theory allows the membership of the elements in the set in binary terms, a bivalent condition- an element either belongs or does not belong to the set. Fuzzy set theory permits the gradual assessment of the membership of elements in a set, described with the aid of a membership function valued in the real unit interval $[0,1]$. Fuzzy sets have been able provide solutions to many real world problems. Fuzzy set theory is an *extension of classical set theory* where elements have degrees of membership.

Generally, several systems are mostly related to uncertainty and inaccuracy. The problem of inaccuracy is considered in general an exact science and that of uncertainty is considered as vague or fuzzy and accidental. The definition given here generalizes that of Aumann [1] for set-valued mappings. A differential and integral calculus for fuzzy-set-valued, shortly fuzzy-valued, mappings was developed in recent papers of Dubois and Prade [5, 6, 7] and Puri and Ralescu [18]. For fuzzy concepts, recently Diamand and Kloeden [4] established the theory of metric space of fuzzy sets. In particular, Kaleva [9] researched the fuzzy differential equations, Cauchy problem for continuous fuzzy differential equations was studied

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by Nieto [14], and Song et al. [16] obtained the global solutions. Park and Han [17] studied the existence and uniqueness theorem for a solution of fuzzy Volterra integral equations by using the method of successive approximation. Seikkala [15] proved the existence and uniqueness of the fuzzy solution for the following systems:

$$\begin{aligned} u'(t) &= f(t,u(t)), \qquad t \in J = [0,b], \\ u(0) &= u_0, \end{aligned}$$

where f is a continuous mapping from $R^+ \times R$ into R and x_0 is a fuzzy number. Recently, the above concept has been extended to the integrodifferential equations by Balasubramaniam and Muralisankar [2]. Ding and Kandel [8] analyzed a way to combine differential equations with fuzzy sets to form a fuzzy logic systems called a fuzzy dynamical system, which can be regarded to form a fuzzy neutral functional differential equations. During the past decades, the impulsive differential equations have attracted many authors since it is much richer than the corresponding theory of differential equations [2, 10, 20]. Among the previous research, little is concerned with integrodifferential equations with impulsive conditions and nonlocal conditions. Here, motivated by[11, 12, 19], we will combine these earlier work and extend the study to the following impulsive nonlinear fuzzy integrodifferential equations with nonlocal conditions:

$$\begin{aligned} \frac{d}{dt}(u(t) - g(t,u(t))) &= a(t)u(t) + \int_0^t k(t,s,u(s))ds \\ &\quad + f(t,u(t)), \qquad t \in [0,b], \\ u(0) + h(u) &= u_0, \\ \Delta u(t_k) &= I_k(u(t_k^-)) \end{aligned}$$

where $a : J \to E_N$ is a fuzzy coefficient, E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α -evel intervals, $g : J \times E_N \to E_N$, $f : J \times E_N \to E_N$, $h : E_N \to E_N$ and $k : J \times J \times E_N \to E_N$ are nonlinear continuous functions, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $I_k \in C(E_N, E_N)$ are continuous functions.

2. PRELIMINARIES

A fuzzy subset of R^n is defined in terms of a membership function which assigns to each point $x \in \mathrm{R}^n$ a grade of membership in the fuzzy set. Such a membership function is denoted by

$$u : \mathrm{R}^n \to [0,1].$$

Throughout this paper, we assume that u maps R^n onto $[0,1]$, $[u]^0$ is a bounded subset of R^n, u is upper semicontinuous, and u is fuzzy convex. We denote by E^n the space of all fuzzy subsets u of R^n which are normal, fuzzy convex, and upper semicontinuous fuzzy sets with bounded supports. In particular, E^1 denotes the space of all fuzzy subsets u of $\mathbb{R}$.

A fuzzy number a in real line $\mathbb{R}$ is a fuzzy set characterized by a membership function χ_a, $\chi_a : \mathbb{R} \to [0,1]$. A fuzzy number a is expressed as $a = \int_{x \in \mathbb{R}} \frac{\chi_a}{x}$ with the understanding that $\chi_a(x) \in [0,1]$, represents the grade of membership of x in a and $\int$ denotes the union of $\frac{\chi_a}{x}$.

Definition 2.1 *A fuzzy number* $a \in \mathbb{R}$ *is said to be convex if, for any real numbers* x, y, z *in* $\mathbb{R}$ *with* $x \le y \le z$,

$$\chi_a(y) \ge \min\{\chi_a(x), \chi_a(z)\}$$

Definition 2.2 *The height of a fuzzy set is the largest membership value attained by any point.*

Definition 2.3 *If the height of a fuzzy set equals one, then the fuzzy set is called normal. Thus, a fuzzy number* $a \in \mathbb{R}$ *is called normal, if the followings holds:* $\max_x \chi_a(x) = 1$.

Result 2.1 *Let* E_N *be the set of all upper semicontinuous convex normal fuzzy numbers with bounded* α *- level intervals (see [13]). This means that if* $a \in E_N$, *then* $\alpha -$ *level set*

$$[a]^\alpha = \{x \in \mathbb{R} : a(x) \ge \alpha, 0 \le \alpha \le 1\},$$

is a closed bounded interval, which we denote by

$$[a]^\alpha = [a_q^\alpha, a_r^\alpha]$$

and there exists a $t_0 \in \mathbb{R}$ such that $a(t_0) = 1$.

Result 2.2 *Two fuzzy numbers* a *and* b *are called equal* $a = b$, *if* $\chi_a(x) = \chi_b(x)$, *for all* $x \in \mathbb{R}$. *It follows that*

$$a = b \Leftrightarrow [a]^\alpha = [b]^\alpha, \ \text{for all } \alpha \in (0,1].$$

Result 2.3 *A fuzzy number* a *may be decomposed into its level sets through the resolution identity* $a = \int_0^1 \alpha [a]^\alpha$, where $\alpha[a]^\alpha$ is the product of a scalar α with the set $[a]^\alpha$ and $\int$ is the union of $[a]^\alpha$ with α ranging from 0 to 1.

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Definition 2.4 *The support of a fuzzy set* A *in the universal set* U *is a crisp set that contains all the elements of* U *that have non zero membership values in* A *, that is,* $supp(A) = \{x \in U : \chi_a(x) > 0\}$, where $supp\ (A)$ denotes the support of fuzzy set A. Hences the support Γ_a of a fuzzy number a is defined, as a special case of level set, by the following: $\Gamma_a = \{x : \chi_a(x) > 0\}$.

Definition 2.5 *A fuzzy number* $a \in \mathrm{R}$ *is said to be positive if* $0 < a_1 < a_2$ *holds for the support* $\Gamma_a = [a_1, a_2]$ *of a, that is,* Γ_a *is in the positive real line. Similarly,* a *is called negative if* $a_1 \le a_2 < 0$ *and zero if* $a_1 \le 0 \le a_2$.

Lemma: 2.1*[15] If* $a, b \in E_N$ *, then for* $\alpha \in (0,1]$ *,*

$$[a+b]^\alpha = [a_q^\alpha + b_q^\alpha, a_r^\alpha + b_r^\alpha],\ [a-b]^\alpha = [a_q^\alpha - b_q^\alpha, a_r^\alpha - b_r^\alpha].$$

$$[ab]^\alpha = [\min\{a_i^\alpha b_i^\alpha\}, \max\{a_i^\alpha b_i^\alpha\}],\ (i, j = q, r),$$

Lemma: 2.2*[15] Let* $[a_q^\alpha, a_r^\alpha]$ $0 < \alpha \le 1$*, be a given family of nonempty intervals. If* $[a_q^\beta, a_r^\beta] \subset [a_q^\alpha, a_r^\alpha]$ *for* $0 < \alpha \le \beta$, $[\lim_{k\to\infty} a_q^{\alpha_k}, \lim_{k\to\infty} a_r^{\alpha_k}] = [a_q^\alpha, a_r^\alpha]$,

whenever (α_k) is nondecreasing sequence converting to $\alpha \in (0,1]$, then the family $[a_q^\alpha, a_r^\alpha]$, $0<\alpha\le 1$, are the α -level sets of a fuzzy number $a \in E_N$, Let x be a point in R^n and A be a nonempty subsets of R^n. We define the Hausdroff separation of B from A by

$d(x, A) = \inf\{\|x - a\| : a \in A\}$. Now let A and B be nonempty subsets of R^n. We define the Hausdroff separation of B from A by $d_H^*(B, A) = \sup\{d(b, A) : b \in B\}$. In general, $d_H^*(A,B) \ne d_H^*(B,A)$. We define the Hausdroff distance between nonempty subsets of A and B of R^n by $d_H(A,B) = \max\{d_H^*(A,B), d_H^*(B,A)\}$. This is now symmetric in A and B . Consequently, 1. $d_H(A,B) \ge 0\ with d_H(A,B) = 0$ if and only if $\overline{A} = \overline{B}$; 2. $d_H(A,B) = d_H(B,A)$;

3. $d_H(A,B) \le d_H(A,C) + d_H(C,B)$; for any nonempty subsets of A , B and C of R^n. The Hausdroff distance is a metric, the Hausdroff metric. The supremum metric d_∞ on E^n is defined by

$$d_\infty(u,v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in (0,1]\},\ \ for all\ u, v \in E^n$$

and is obviously metric on E^n. The supremum metric H_1 on $C(J, E^n)$ is defined by

$$H_1(x, y) = \sup\{d_\infty(x(t), y(t) : t \in J),\ for\ all\ x, y \in \mathrm{C}(J : E^n)\}.$$

3.EXISTENCE AND UNIQUENESS OF FUZZY SOLUTION

In this section, we consider the existence and uniqueness of the fuzzy solution for impulsive nonlinear fuzzy integrodifferential equations with nonlocal conditions:

$$\frac{d}{dt}(u(t) - g(t,u(t))) = a(t)u(t) + \int_0^t k(t,s,u(s))\,ds + f(t,u(t)), \quad t \in [0,b],$$
$$u(0) + h(u) = u_0,$$
$$\Delta u(t_k) = I_k(u(t_k^-)) \qquad (1)$$

where $a : J \to E_N$ is a fuzzy coefficient, E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α - level intervals, and are nonlinear continuous functions, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $I_k \in C(E_N, E_N)$ are continuous functions. and satisfy a global Lipschitz condition, that is, there exist a positive constant $l_g, l_{g_1}, l_f, l_k, l_h$ and l_I, for all $u_1(t), u_2(t) \in E_N$. Let I be an interval. A mapping $u : I \to E_N$ of a fuzzy process u, then

$$[u'(t)]^\alpha = [(u_q^\alpha)', (u_r^\alpha)'], \quad 0 < \alpha \leq 1.$$

The fuzzy integral $\int_a^b u(t)dt, \quad a, b \in I,$ is defined by $\int_a^b [u(t)]^\alpha = [\int_a^b u_q^\alpha, \int_a^b u_r^\alpha]$ provided Lebesgue integrals on the right exist.

Theorem 3.1 Let f, g, I_k and k satisfy a global Lipschitz condition, for every $u_0 \epsilon E_N$ then the nonlinear impulsive fuzzy integrodifferential equation with nonlocal condition (1) has a unique solution $u \in C(J, E_N)$.

Proof. For each $u(t) \in E_N, t \in J$,

$$\begin{aligned}\mathcal{F}_0 u(t) &= S(t)[u_0 - h(u) - g(0, u(0) - h(u))] + g(t, u(t)) + \int_0^t S(t-s) g(s, u(s)) ds \\ &+ \int_0^t S(t-s)\Big(\int_0^s k(s, r, u(r)) dr\Big) ds + \int_0^t S(t-s) f(s, u(s)) ds + \sum_{0<t_k<t} S(t-t_k) I_k(u(t_k^-)),\end{aligned}$$

where $S(t)$ is a fuzzy number and

$$[S(t)]^\alpha = [S_q^\alpha(t), S_r^\alpha(t)] = [exp\{\int_0^t a_q^\alpha(s)\}, exp\{\int_0^t a_r^\alpha(s)\}],$$

and $S_i^\alpha(t)(i = q, r)$ is continuous. That is, there exist a constant $l_s > 0$ such that $|S_i^\alpha(t)| \leq l_s$, for all $t \in J$. Thus $F_0 u : J \to E_N$ is continuous, $F_0 : C(J, E_N) \to C(J, E_N)$. For $u_1, u_2 \in C(J, E_N)$, we have

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$$
\begin{aligned}
& d_H\Big([(\mathcal{F}_0(u_1)(t))]^\alpha, [(\mathcal{F}_0(u_2)(t))]^\alpha\Big) \\
&= d_H\Big(\Big[S(t)(u_0 - h(u_1) - g(0, u_0 - h(u_1))) + g(t, u_1(t)) + \int_0^t S(t-s)g(s, u_1(s))ds \\
&\quad + \int_0^t S(t-s)(\int_0^s k(s,r,u_1(r))dr)ds + \int_0^t S(t-s)f(s,u_1(s))ds + \sum_{0<t_k<t} S(t-t_k)I_k(u_1(t_k^-))\Big]^\alpha, \\
&\quad \Big[S(t)(u_0 - h(u_2) - g(0, u_0 - h(u_2))) + g(t, u_2(t)) + \int_0^t S(t-s)g(s, u_2(s))ds \\
&\quad + \int_0^t S(t-s)(\int_0^s k(s,r,u_2(r))dr)ds + \int_0^t S(t-s)f(s,u_2(s))ds + \sum_{0<t_k<t} S(t-t_k)I_k(u_2(t_k^-))\Big]^\alpha\Big)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & l_s l_h d_H([u_1]^\alpha, [u_2]^\alpha) + (l_s(l_g + l_{g_1 l_h}) + l_g) d_H([u_1]^\alpha, [u_2]^\alpha) + l_g d_H([u_1]^\alpha, [u_2]^\alpha) \\
& + l_s l_k b \int_0^t d_H([u_1]^\alpha, [u_2]^\alpha)ds + l_s l_f \int_0^t d_H([u_1]^\alpha, [u_2]^\alpha)ds \\
& + l_s l_I d_H([u_1]^\alpha, [u_2]^\alpha)
\end{aligned}
$$

Let, $L_1 = l_s(l_h + l_I + l_g + l_{g_1} l_h) + l_g, L_2 = l_s(l_k b + l_f)$. Therefore

$$
d_\infty((\mathcal{F}_0 u_1, \mathcal{F}_0 u_2) = \sup_{t \in J} d_H([(\mathcal{F}_0 u_1)]^\alpha, [(\mathcal{F}_0 u_2)]^\alpha)
$$

$$
\leq L_1 d_\infty([u_1]^\alpha, [u_2]^\alpha) + L_2 \int_0^t d_\infty([u_1]^\alpha, [u_2]^\alpha)ds.
$$

Hence

$$
H_1(\mathcal{F}_0 u_1, \mathcal{F}_0 u_2) = \sup_{t \in J} d_\infty(\mathcal{F}_0 u_1, \mathcal{F}_0 u_2)
$$

$$
\leq (L_1 + L_2 b) H_1(u_1, u_2).
$$

we take sufficiently small b , $(L_1 + L_2 b) < 1$. Hence, F_0 is a contraction mapping. By the Banach fixed point theorem,implusive fuzzy integrodifferential equation with nonlocal condition has a unique fixed point $u \in C(J, E_N)$.

4.EXAMPLE

Consider the fuzzy solution of the nonlinear impulsive fuzzy integrodifferential equation of the form:

$$
\begin{aligned}
\frac{d}{dt}(u(t) - \mathbf{2}tu(t)^2) &= \mathbf{2}u(t) + \mathbf{2}tu(t)^2 + \mathbf{2}tu(t)^2, \quad t \in J, \\
u(0) &= \sum_{k=1}^{p} u(t_k) \\
\Delta u(t_k) &= I_k(u(t_k)) = \frac{1}{1 + u(t_k)}
\end{aligned}
\tag{2}
$$

The α - level set of fuzzy number **2** is $[2]^\alpha = [\alpha+1, 3-\alpha],\ for\ \alpha \in [0,1]$ Therefore,f, g, h, I_k are satisfies the global lipschitz conditions and choose b is sufficiently small. Then all conditions stated in Theorem3.1 are satisfied, so the problem has a unique fuzzy solution.

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