IDEAL STRUCTURE IN 3-RINGS

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Abstract: This paper presents Ideal structure in 3- rings, principal ideal of a 3- ring is generated by an idempotent, every principal ideal has unique inverse principal ideal and the class of all principal ideals of a 3- ring is a complemented modular lattice.

Definition 1: A commutative ring (R, +, ., 1) such that $x^3 = x$, $3x = 0$ for all x in R is called a 3-ring.

Definition 2: An ideal in a 3-ring R is a set I of R such that

(a) $x, y \in I \Rightarrow x + y \in I$

(b) $x \in I, y \in R \Rightarrow xy \in I$

Here after R stands for a 3-ring.

Note: Intersection of ideals is an ideal.

Note: (a) The class of all ideals of R forms a partially ordered set with respect to the set theoretical inclusion. This set has a minimum element O = (O) = OR, and maximal element R = (1).

(b) Suppose I, J, … are ideals of R, then there exists a maximal ideal which is contained in I, J, …, it is denoted by glb{I, J, …}.

(c) Suppose I, J, … are ideals. Then there exists a minimal ideal which contains I, J, … and is denoted by lub{I, J, …}.

(d) I, J are ideals, then

$I \vee J = \text{lub}\{I, J\}$

$I \wedge J = \text{glb}\{I, J\}$

$\therefore$ The set of all ideals of R forms a lattice with meet $I \wedge J$ and join $I \vee J$, zero(0), and unit R = (1).

Two ideals I, J are inverses if $I \wedge J = O$, $I \vee J = R$.

$I \vee J = \{x+y \mid x \in I, y \in J\}$.

Note: (1) Here after R stands for 3-ring with unit 1.

(2) For every $a \in R$, aR denoted (a) is an ideal called principal ideal.

Definition 3: An element $e \in R$ is called an idempotent if $e^2=e$.

Lemma 1: e is an idempotent $\Leftrightarrow$ 1 – e is an idempotent.

Proof: Suppose e is an idempotent.

ISBN - 978-93-81583-56-2

$(1-e)^2 = 1+e^2 - 2e = 1+e-2e = 1-e$
$\therefore$ 1 – e is an idempotent.

Conversely suppose 1-e is an idempotent, then 1-(1-e) is an idempotent, i.e., e is an idempotent.

Note that if e is an idempotent then e(1-e) = 0.

Lemma 2: For any $a \in R$, a^2 is an idempotent.

Proof: Suppose $a \in R \Rightarrow a^3 = a$ ($\because$ R is a 3-ring).

$(a^2)^2 = a^2 . a^2 = a^3 . a = a . a = a^2$
$\therefore$ a^2 is an idempotent for every $a \in R$.

Lemma 3: For every $a \in R$, (a) = (e) where e is an idempotent.
Proof: Suppose $a \in R$, put $e = a^2$.
Then e is an idempotent.

<u>**Claim:**</u> (a) = (e).
$x \in (a) \Rightarrow x = az$ for some $z \in R$
$= a^3z = a^2 (az) \in (a^2) = (e)$.
$\therefore x \in (e)$.
$x \in (a^2) \Rightarrow x = a^2z$ for some $z \in R$
$= a(az) \in (a)$.
$\therefore (a^2) \subseteq (a)$ i.e., $(e) \subseteq (a)$.
$\therefore$ (a) = (e) where e is an idempotent.

Note: $x \in (e)$ where e is an idempotent $\Leftrightarrow x = ex$.
For: $x \in (e) \Rightarrow x = ez$ for some $z \in R$.
$ex = e(ez) = e^2z = ez = x$.
$\therefore x = ex$.
$\therefore x \in (e) \Rightarrow x = ex$

Suppose $x = ex \Rightarrow x \in (e)$
$\therefore x \in (e) \Leftrightarrow x = ex$.

Lemma 4: For every $a \in R$ there is an ideal J such that (a) and J are inverse ideals.

Proof: Suppose $a \in R$.

$\Rightarrow \exists$ an idempotent $e \in R$ such that (a) = (e).
Put I = (e).
Consider J = (1-e).
Then $I \wedge J = 0$, $I \vee J = R$.

$\therefore$ I and J are inverse ideals.

ISBN - 978-93-81583-56-2

For $x \in I \wedge J \Rightarrow x \in I, x \in J$
$\Rightarrow x = ey, x = (1\text{-}e)z$ for some $y, z \in R$.
$1 = e + [1 - e] \in I \vee J$
$\therefore I \vee J = R$.
$x \in I \wedge J \Rightarrow x \in I, x \in J$
$\Rightarrow x = ex,\ x = (1\text{-}e)\ x$
$x = (1\text{-}e)x = x - ex = x - x = 0$.
$\therefore I \wedge J = 0.4$

$\therefore$ I, J are inverse ideals.

Lemma 5: I and J are inverse ideals. Then there is an idempotent $e \in R$ such that $I = (e), J = (1\text{-}e)$.

Proof: Since $I \vee J = R$, so $1 \in I \vee J$
$\Rightarrow 1 = x + y$ for some $x \in I, y \in J$.
Let $z \in I$.
$1 = x + y$
$z = xz + yz \Rightarrow z - xz = yz \in I$.
But $yz \in J \Rightarrow yz = 0$
$\because I \wedge J = 0$.
$z = xz + yz = xz \in (x) \Rightarrow I \subseteq (x)$ and clearly $(x) \subseteq I$.
$\therefore I = (x)$.

Similarly $J = (1\text{-}x)$
$\because y = 1\text{-}x$

Consider $x - x^2 = x\ (1\text{-}x) = xy = yx$.
$\therefore x - x^2 \in (x), (y)$
$\therefore x - x^2 \in I \cap J$
$\Rightarrow x - x^2 = 0$
$\therefore x^2 = 0$.

x is an idempotent.
Take $x = e$.
$\therefore I = (e), J = (1 - e)$.

Lemma 6: I and J are inverse ideals and $I = (e)$ where $e^2 = e$ as in the above lemma, then e is unique idempotent.

Proof: $I = (e)$, then $J = (1 - e)$.
Suppose there exist another idempotent f such that $I = (f)$,
$J = (1 - f) \Rightarrow (e) = (f), (1 - e) = (1 - f)$.

<u>Claim</u>: $e = f$.
$\because e \in (e) \Rightarrow e \in (f)$

$\Rightarrow e = ef.$
Similarly f = fe.
$\therefore e = ef = fe = f.$
$\therefore$ e is unique.

Definition 4: The ring R is said to be semi-simple, if the radical of R consists of the zero element alone.

Theorem 1: Every 3-ring is a semi-simple ring.

Proof: Suppose R is a 3-ring.

$\therefore$ For every $a \in R$, there exists an idempotent e such that (a) = (e).

Let I be a non-zero ideal.
Let $a \in I$ and $a \neq 0$
$\Rightarrow$ there exists an idempotent e such that (a) = (e)
$\Rightarrow e \neq 0$
$\Rightarrow e = e^n \in I^n$
$\Rightarrow$ I is not nilpotent ideal.
$\therefore$ R is a semi-simple ring.

Theorem 2: If I and J are two principal ideals, then there exist two idempotents e, f such that ef = 0 and $I \vee J = (e) \vee (f)$.

Proof: Suppose R is a 3-ring and I is a principal ideal, then there exist an idempotent $e \in R$ such that I = (e).

Let J= (b).
Let $J_1 = ((1\text{-}e)b)$
$I \vee J = \{x+y | x \in I, y \in J\}$
$= \{eu + bv \mid u, v \in R\}$
$I \vee J_1 = \{ x+y \mid x \in I, y \in J_1\}$
$= \{eu' + (1\text{-}e)bv \mid u', v \in R\}$
$= e (u' - bv) + bv$
$= I \vee J$
$\because J_1$ is a principal ideal and R is a 3-ring, then there exist an idempotent f_1 such that $J_1 = (f_1)$.
$\because f_1 \in J_1 \Rightarrow f_1 = (1\text{-}e)bw$ for some $w \in R$.
$e f_1 = e(1\text{-}e)bw = 0.$
$\therefore e f_1 = 0.$
Let $f = f_1(1\text{-}e)$
$f f_1 = f_1(1\text{-}e) f_1 \qquad = (f_1 - ef_1) f_1$
$= f_1 . f_1 = 0.$

$\therefore f f_1 = f_1$.
$f^2 = f . f \quad = f f_1(1\text{-}e)$
$\quad = f_1 (1\text{-}e) = f.$
$\therefore f^2 = f.$

$\therefore$ f is an idempotent.
$\therefore I \vee J = I \vee J_1 \qquad = (e) \vee (f_1) = (e) \vee (f).$
$\therefore I \vee J = (e) \vee (\tilde{f})$
$ef = e f_1 (1\text{-}e) = 0$
$\therefore ef = 0.$

Lemma 7: If I and J are principal ideals then $I \vee J$ is a principal ideal.

Proof: From above lemma, there exist idempotents e, f such that $I \vee J = (e) \vee (f)$ and $ef = 0$.

$\because e + f \in (e) \vee (f)$
$\Rightarrow (e + f) \subseteq (e) \vee (f)$
$\because e \in (e + f),\ f \in (e + f) \Rightarrow (e) \subseteq (e + f), (f) \subseteq (e + f).$
$\therefore (e) \vee (f) \subseteq (e + f)$
$I \vee J = (e) \vee (f)$
$\quad = (e + f)$
$\therefore I \vee J = (e + f).$

Theorem 3: The class of all principal ideals of a 3-ring is a complemented modular lattice.

Proof: For every pair I, J of principal ideals, $I \vee J$, $I \wedge J$ are principal ideals.
And (0), R = (1) are principal ideals.

Suppose I is a principal ideal of R. Then there exist a principal ideal J such that $I \vee J = R$, $I \wedge J = 0$.

Let I, J, K be principal ideals and $I \subseteq K$.

<u>Claim</u>: $(I \vee J) \wedge K = I \vee (J \wedge K)$
Consider,
$(I \vee J) \wedge K \supseteq I \wedge J = I$ and $(I \vee J) \wedge K \supseteq J \wedge K$
$\therefore (I \vee J) \wedge K \supseteq I \vee (J \wedge K)$
Let $x \in (I \vee J) \wedge K$
$\Rightarrow x \in I \vee J, x \in K$
$\Rightarrow x = y + z$, where $y \in I, z \in J$.
$\because x \in K, y \in K\ (\because I \subseteq K)$
$\Rightarrow y - x \in K$
$\therefore z \in K.$

$\therefore z \in J \cap K$
$\therefore y + z \in I \vee (J \cap K)$
$\Rightarrow x \in I \vee (J \wedge K)$
$\therefore (I \vee J) \wedge K = I \vee (J \wedge K)$.

$\therefore$ The class of all principal ideals of R is a complemented modular lattice.

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