# SUM OF d-POWERS OF GENERALISED FIBONACCI LIKE POLYNOMIALS

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***Abstract : In this paper the sum of d-powers of Generalised Fibonacci like Polynomials $V_n(x) = xV_{n-1}(x) + V_{n-2}(x)$ ; $n \geq 3$ with $V_1(x) = p$ and $V_2(x) = x$, where $p \geq 3$ is found.***

***Keywords : Fibonacci Polynomials, Lucas Polynomials, Pell Polynomials and Pell-Lucas Polynomials.***

## 1. INTRODUCTION

The generalized Fibonacci like polynomials are defined by the recurrence relation

$V_n(x) = xV_{n-1}(x) + V_{n-2}(x)$ ; $n \geq 3$ with
$V_1(x) = a$ and $V_2(x) = bx$,

First few polynomials are

$V_3(x) = bx^2 + a$
$V_4(x) = (x^3 + x)b + ax$
$V_5(x) = (x^4 + 2x^2)b + (x^2 + 1)a$
$V_6(x) = (x^5 + 3x^3 + x)b + (x^3 + 2x)a$
………………………

If a = b = 1 then
$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ ; with $F_1(x) = 1$, $F_2(x) = x$
(Fibonacci Polynomials)

If a = 2, b = 1 then
$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x)$ ; with $L_0(x) = 2$, $L_1(x) = x$
(Lucas Polynomials)

If a = 1, b = 2 then
$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$ ; with $P_1(x) = 1$, $P_2(x) = 2x$
(Pell Polynomials)

If a = b = 2 then
$Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x)$ ; with $Q_0(x) = 2$, $Q_1(x) = 2x$
(Pell-Lucas Polynomials)

If a = p, p ≥ 3 and b = 1 then the generalized Fibonacci like Polynomials are known as Mathematics Year 2012 polynomials and denoted $MY_n^P(x)$ or $M_n^P(x)$

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The polynomials $MY_n^P(x)$ are defined by the recurrence relation

$MY_n^P(x) = xM_{n-1}^P(x) + M_{n-2}^P(x)$, $p \geq 3$ with $M_1^P(x) = p$, $M_2^P(x) = x$

In this paper the sum of d-powers of Lucas polynomials and $MY_n^P(x)$ polynomials are obtained.

## 2. RELATION BETWEEN $F_N(X)$, $L_N(X)$ AND $M_n^P(x)$

$MY_n^P(x) = L_n(x) + (p-2)F_{n-1}(x)$ with

$M_1^P(x) = p$, $M_2^P(x) = x$, $p = 3, 4, 5, \ldots$ and $n \geq 3$.

when p = 1, $M_n^1(x)$ is the sequence of Fibonacci polynomials.

when p = 2, $M_n^2(x)$ is the sequence of Lucas polynomials.

## 3. CLOSED FORM EXPRESSIONS OF THE POLYNOMIALS $M_n^P(x)$

The recurrence relation of $M_n^P(x)$ is given by

$$M_n^P(x) = xM_{n-1}^P(x) + M_{n-2}^P(x) \qquad \ldots \qquad (1)$$

$n \geq 3$ with $M_1^P = p$, $M_2^P = x$, $p \geq 3, 4, 5, \ldots$

when p = 3, equation (1) takes the form

$$M_n^3(x) = xM_{n-1}^3(x) + M_{n-2}^3(x)$$

We use the notation $M_n(x)$ for $M_n^3(x)$

The homogenous equation of (1) is

$$t^2 - xt - 1 = 0 \qquad \ldots \qquad (2)$$

Solving equation (2) for t

$$t = \frac{x \pm \sqrt{x^2 + 4}}{2}$$

The solutions of (1) can be denoted by

$$\alpha(x) = \frac{x + D}{2} \text{ and } \beta(x) = \frac{x - D}{2}$$

where $D = \sqrt{x^2 + 4}$

The general solution is given by

$M_n(x) = C_1\alpha(x)^n + C_2\beta(x)^n$, $n \geq 1$

The constants $C_1$ and $C_2$ can be found using the initial conditions

Now $M_1(x) = C_1\alpha(x) + C_2\beta(x)$

$M_2(x) = C_1\alpha(x)^2 + C_2\beta(x)^2$

Using the initial conditions, we get

$$C_1\alpha(x) + C_2\beta(x) = 3 \quad \ldots \quad (3)$$

$$C_1\alpha(x)^2 + C_2\beta(x)^2 = x \quad \ldots \quad (4)$$

Multiplying equation (3) by $\alpha(x)$ and subtracting from (4), we get

$$C_2 = \frac{3\alpha(x) - x}{\beta(x)[\alpha(x) - \beta(x)]}$$

Substituting the value of $C_2$ in (3) we get

$$C_1 = \frac{x - 3\beta(x)}{\alpha(x)[\alpha(x) - \beta(x)]}$$

Therefore the general solution of $M_n(x)$ is given by

$$M_n(x) = \left(\frac{x - 3\beta(x)}{\alpha(x) - \beta(x)}\right)\alpha(x)^{n-1} + \left(\frac{3\alpha(x) - x}{\alpha(x) - \beta(x)}\right)\beta(x)^{n-1}$$

Substituting $\alpha(x) = \frac{x + D}{2}$ and $\beta(x) = \frac{x - D}{2}$

$M_n(x)$ can be rewritten as

$$M_n(x) = \left[\left(\frac{3D - x}{2}\right)\alpha(x)^{n-1} + \left(\frac{3D + x}{2}\right)\beta(x)^{n-1}\right]\left(\frac{1}{\alpha(x) - \beta(x)}\right)$$

$$\left(\text{Since } x - 3\beta = \frac{3D - x}{2} \ ; 3\alpha - x = \frac{3D + x}{2}\right)$$

Now $\alpha(x) - \beta(x) = \frac{x}{2} + \frac{D}{2} - \frac{x}{2} + \frac{D}{2} = D$

Hence $M_n(x) = \left(\frac{3D - x}{2D}\right)\alpha(x)^{n-1} + \left(\frac{3D + x}{2D}\right)\beta(x)^{n-1}$

## 4. SUM OF D-POWERS OF $M_n^P(x)$ POLYNOMIALS

Let $M_d^P(x) = \sum_{K=0}^{n} (M^p)^d(K)$

when p = 2, $M_n^2$ polynomials are Lucas polynomials L(n).

First we find, sum of d powers of Lucas polynomials

Let $L_d(n) = \sum_{K=0}^{n} L^d(K)$

Now we write $L^d(K)$ as below

$$L^d(K) = \sum_{i=0}^{d}\binom{d}{i}\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{K-1}$$

$$= \sum_{i=0}^{d}\binom{d}{i}\sum_{K=0}^{n}\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{K-1}$$

$$L^d(K) = \sum_{i=0}^{d}\binom{d}{i}\left[\frac{1-\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{K-1}}{1-\alpha(x)^i\ \beta(x)^{d-i}}\right]$$

Next we find the sum of d-powers of $M_n^3(x)$

Now $$M_n^3(x) = \left(\frac{3D - x}{2D}\right)\alpha(x)^{n-1} + \left(\frac{3D + x}{2D}\right)\beta(x)^{n-1}$$

Let $$M_d(x) = \sum_{K=0}^{n} M^d(K) \quad \ldots \quad (*)$$

Now

$$M^d(K) = \frac{1}{(2D)^d}\sum_{i=0}^{d}\binom{d}{i}\left[(3D + x)^{d-i}(3D - x)^i\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{K-1}\right]$$

Substituting $M^d(K)$ in equation (*)

$$M_d(x) = \sum_{K=0}^{n}\left[\frac{1}{(2D)^d}\sum_{i=0}^{d}\binom{d}{i}\left[(3D + x)^{d-i}(3D - x)^i\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{K-1}\right]\right]$$

$$= \frac{1}{(2D)^d}\sum_{i=0}^{d}\binom{d}{i}\left[(3D + x)^{d-i}(3D - x)^i\sum_{K=0}^{n}\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{K-1}\right]$$

$$= \frac{1}{(2D)^d}\sum_{i=0}^{d}\binom{d}{i}\left[(3D + x)^{d-i}(3D - x)^i\left[\frac{1-\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{n}}{1-\alpha(x)^i\ \beta(x)^{d-i}}\right]\right]$$

Sum of 3rd power of First three Lucas Polynomials

$$L_3(3) = \sum_{i=0}^{3}\binom{d}{i}\sum_{K=0}^{3}\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{K-1}$$

$$= \sum_{i=0}^{3}\binom{d}{i}\left[\frac{1-\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{3}}{1-\alpha(x)^i\ \beta(x)^{d-i}}\right]$$

$$= \binom{3}{0}\frac{1-\left(\alpha(x)^0\ \beta(x)^3\right)^3}{1-\alpha(x)^0\ \beta(x)^3} + \binom{3}{1}\frac{1-\left(\alpha(x)^1\ \beta(x)^2\right)^3}{1-\alpha(x)^1\ \beta(x)^2}$$

$$+\binom{3}{2}\frac{1-\left(\alpha(x)^2\ \beta(x)^1\right)^3}{1-\alpha(x)^2\ \beta(x)^1}+\binom{3}{3}\frac{1-\left(\alpha(x)^3\ \beta(x)^0\right)^3}{1-\alpha(x)^3\ \beta(x)^0}$$

$$=\frac{1-\beta(x)^9}{1-\beta(x)^3}+3\frac{1-\left(\alpha(x)^3\ \beta(x)^6\right)}{1-\alpha(x)^1\ \beta(x)^2}+3\frac{1-\left(\alpha(x)^6\ \beta(x)^3\right)}{1-\alpha(x)^2\ \beta(x)^1}$$

$$+\frac{1-\alpha(x)^9}{1-\alpha(x)^3}+3\frac{1-\left(\alpha(x)^6\ \beta(x)^3\right)}{1-\alpha(x)^2\ \beta(x)^1}+\frac{1-\alpha(x)^9}{1-\alpha(x)^3}$$

$$=1+\beta(x)^3+\left(\beta(x)^3\right)^2+3\left(1+\alpha(x)\beta(x)^2+\left(\alpha(x)\beta(x)^2\right)^2\right)$$

$$+3\left(1+\alpha(x)^2\beta(x)+\left(\alpha(x)^2\beta(x)\right)^2\right)+1+\alpha(x)^3+\left(\alpha(x)^3\right)^2$$

By Direct Method we get

$$L_3(3) = \sum_{K=1}^{3} L^3(K)$$

$$= L^3(1) + L^3(2) + L^3(3)$$

$$= 2^3 + (\alpha(x) + \beta(x))^3 + (\alpha^2(x) + \beta^2(x))^3$$

Let $M_n^3(x) = M_n(x)$

Now the sum of squares of first three terms of $M_n(x)$ is $M_1^2(x) + M_2^2(x) + M_3^2(x)$

Since sum of d power of $M_n(x)$ is

$$M_d(n) = \frac{1}{(2D)^d}\sum_{i=0}^{d}\binom{d}{i}\left[\left(3D+x\right)^{d-i}\left(3D-x\right)^{i}\sum_{K=0}^{n}\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^{K-1}\right]$$

$$=\frac{1}{(2D)^d}\sum_{i=0}^{d}\binom{d}{i}\left[\left(3D+x\right)^{d-i}\left(3D-x\right)^{i}\left[\frac{1-\left(\alpha(x)^i\ \beta(x)^{d-i}\right)^n}{1-\alpha(x)^i\ \beta(x)^{d-i}}\right]\right]$$

We have

$$M_2(3) = \frac{1}{(2D)^2}\sum_{i=0}^{2}\binom{2}{i}\left[\left(3D+x\right)^{2-i}\left(3D-x\right)^{i}\left[\frac{1-\left(\alpha(x)^i\ \beta(x)^{2-i}\right)^3}{1-\alpha(x)^i\ \beta(x)^{2-i}}\right]\right]$$

$$=\frac{1}{(2D)^2}\left[\left(3D+x\right)\frac{1-\beta(x)^6}{1-\beta(x)^2}+2\left(3D+x\right)\left(3D-x\right)\right.$$

$$\left.\left[\frac{1-\left(\alpha(x)\beta(x)\right)^3}{1-\alpha(x)\beta(x)}\right]+\left(3D-x\right)\frac{1-\alpha(x)^6}{1-\alpha(x)^2}\right]$$

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## 5. CONCLUSION

In this paper we have derived formula for sum of d-powers of generalized Fibonacci Like Polynomials when a = p = 3 and b = 1 via their explicit Binet Form. This procedure is extended to find the sum of d-powers of generalized Fibonacci Polynomials when p = 4, 5, etc.

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