A GENERALISED DOUBLE SAMPLING ESTIMATOR OF RATIO (PRODUCT) OF PARAMETERS

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Abstract: In this paper a generalised double sampling estimator representing a class of estimators using information on an auxiliary variable is proposed for the estimation of ratio (product) of population parameters. The bias and mean square error are found, and the properties of the generalised estimator are studied. Classes of estimators depending on optimum values in the sense of minimum mean square error are also investigated.

Keywords: Double sampling, auxiliary variable, bias, mean square error.

1. INTRODUCTION

Let a first phase large simple random sample of size n' be drawn from a population of size N and only the auxiliary variable x_2 be observed on this first phase sample, and further, let both the study variables (y, x_1) and the auxiliary variable x_2 observed on the second phase simple random sample of size n from the first phase sample of size n'. For the population values $\{(Y_i, X_{1i}, X_{2i}); i = 1,2,...,N\}$ on (y, x_1, x_2), let the population means, population variances and population coefficient of variation of (y, x_1, x_2) be $(\bar{Y}, \bar{X}_1, \bar{X}_2)$, $(S_y^2, S_{x_1}^2, S_{x_2}^2)$ and (C_0, C_1, C_2) respectively. The population ratio (R) of the population means $\bar{Y}$ and $\bar{X}_1$, and their product (P) are $R = \bar{Y}/\bar{X}_1$ and $P = \bar{Y}.\bar{X}_1$ respectively. Further let ρ_{01}, ρ_{02} and ρ_{12} be the correlation coefficient between (y, x_1), (y, x_2) and (x_1, x_2) respectively. Further $C = \rho_{02}\frac{C_0}{C_2} - \rho_{12}\frac{C_1}{C_2}$, $\lambda_{02} = \frac{\mu_{102}}{\bar{Y}\mu_{002}}$ and $\lambda_{12} = \frac{\mu_{012}}{\bar{X}_1\mu_{002}}$. Also, for first phase sample values $(x_{2i}; i = 1,2,...,n')$ on x_2, the sample mean and sample variance are given by $\bar{x}_2'$ and $s_{x_2}'^2$ respectively. For second phase sample values $\{(y_i, x_{1i}, x_{2i}); i = 1,2,...,n\}$ on (y, x_1, x_2) the sample means and sample variances are given by $(\bar{y}, \bar{x}_1, \bar{x}_2)$ and $(s_y^2, s_{x_2}^2, s_{x_2}^2)$ respectively.

Using auxiliary information on x_2, the double sampling estimators of Singh (1965) for the ratio $R = \bar{Y}/\bar{X}_1$ and the product $P = \bar{Y}.\bar{X}_1$ are

$$\hat{R}_{1d} = \frac{\bar{y}}{\bar{x}_1}\frac{\bar{x}_2}{\bar{x}_2'} = \hat{R}\frac{\bar{x}_2}{\bar{x}_2'} \qquad \hat{R}_{2d} = \frac{\bar{y}}{\bar{x}_1}\frac{\bar{x}_2'}{\bar{x}_2} = \hat{R}\frac{\bar{x}_2'}{\bar{x}_2}$$

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$\hat{P}_{1d} = \bar{y}.\bar{x}_1 \dfrac{\bar{x}_2}{\bar{x}_2^{'}} = \hat{P}\dfrac{\bar{x}_2}{\bar{x}_2^{'}}$ $\qquad$ $\hat{P}_{2d} = \bar{y}.\bar{x}_1 \dfrac{\bar{x}_2^{'}}{\bar{x}_2} = \hat{P}\dfrac{\bar{x}_2^{'}}{\bar{x}_2}$

where $\hat{R} = \dfrac{\bar{y}}{\bar{x}_1}$ and $\hat{P} = \bar{y}.\bar{x}_1$ respectively.

For estimating R , the proposed generalised double sampling estimator is

$$\hat{R}_g = g(\hat{R}, \bar{x}_2, \bar{x}_2^{'}, s_{x_2}^2, s_{x_2}^{'2}) = g(t) \tag{1.1}$$

and $g(t)$ satisfies the validity conditions of Taylor's series expansion is a bounded function of $t = (\hat{R}, \bar{x}_2, \bar{x}_2^{'}, s_{x_2}^2, s_{x_2}^{'2})$ such that at the point $T = (R, \bar{X}_2, \bar{X}_2, S_{x_2}^2, S_{x_2}^2)$

$$g(t = T) = R \tag{1.2}$$

$$g_0 = 1,\ g_1 = -g_2,\ g_3 = -g_4,\ g_{00} = 0,$$

$$g_{01} = -g_{02},\ g_{03} = -g_{04} \tag{1.3}$$

where first order partial derivatives are $g_0 = \left.\dfrac{\partial g}{\partial \hat{R}}\right|_T$, $g_1 = \left.\dfrac{\partial g}{\partial \bar{x}_2}\right|_T$, $g_2 = \left.\dfrac{\partial g}{\partial \bar{x}_2^{'}}\right|_T$,

$g_3 = \left.\dfrac{\partial g}{\partial s_{x_2}^2}\right|_T$ and $g_4 = \left.\dfrac{\partial g}{\partial s_{x_2}^{'2}}\right|_T$ of $g(.)$ w r t $\hat{R}$, $\bar{x}_2$, $\bar{x}_2^{'}$, $s_{x_2}^2$ and $s_{x_2}^{'2}$ respectively at the point $t = T$ and the second partial derivatives are $g_{00} = \left.\dfrac{\partial^2 g}{\partial \hat{R}^2}\right|_T$, $g_{01} = \left.\dfrac{\partial^2 g}{\partial \hat{R} \partial \bar{x}_2}\right|_T$,

$g_{02} = \left.\dfrac{\partial^2 g}{\partial \hat{R} \partial \bar{x}_2^{'}}\right|_T$, $g_{03} = \left.\dfrac{\partial^2 g}{\partial \hat{R} \partial s_{x_2}^2}\right|_T$ and $g_{04} = \left.\dfrac{\partial^2 g}{\partial \hat{R} \partial s_{x_2}^{'2}}\right|_T$ of $g(.)$ w r t $\hat{R}$, $\left(\hat{R}, \bar{x}_2\right)$, $\left(\hat{R}, \bar{x}_2^{'}\right)$, $\left(\hat{R}, s_{x_2}^2\right)$ and $\left(\hat{R}, s_{x_2}^{'2}\right)$ respectively at the point $t = T$.

Some particular members belonging to the generalised double sampling estimator $\hat{R}_g$ are $\hat{R}_1 = \hat{R}\dfrac{\bar{x}_2}{\bar{x}_2^{'}}\dfrac{s_{x_2}^2}{s_{x_2}^{'2}}$, $\hat{R}_2 = \hat{R}\dfrac{\bar{x}_2^{'}}{\bar{x}_2}\dfrac{s_{x_2}^{'2}}{s_{x_2}^2}$, $\hat{R}_3 = \hat{R} + k_1(\bar{x}_2 - \bar{x}_2^{'}) + k_2(s_{x_2}^2 - s_{x_2}^{'2})$, and $\hat{R}_4 = \hat{R}\left(\dfrac{\bar{x}_2}{\bar{x}_2^{'}}\right)^{k_1}\left(\dfrac{s_{x_2}^2}{s_{x_2}^{'2}}\right)^{k_2}$ and $\hat{R}_5 = (1 - k_1)\hat{R} + k_1\hat{R}\left(\dfrac{\bar{x}_2}{\bar{x}_2^{'}}\right)\left(\dfrac{s_{x_2}^2}{s_{x_2}^{'2}}\right)$ where k_1 and k_2 are characterizing scalars to be chosen suitably, as the validity conditions (1.2) and (1.3) are satisfied. Also, some more double sampling estimators can be seen in the literature which satisfy the validity conditions (1.2) and (1.3) [see Cochran (1977), Sukhatme et. al. (1984) and Murthy (1967) for further details].

2. BIAS AND MEAN SQUARE ERROR

Let $\mu_{rsl} = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^r (X_{1i} - \bar{X}_1)^s (X_{2i} - \bar{X}_2)^l$ for $r, s, l = 0, 1, 2, 3, 4$; $\bar{y} = \bar{Y}(1+e_0)$, $\bar{x}_1 = \bar{X}_1(1+e_1)$, $\bar{x}_2 = \bar{X}_2(1+e_2)$, $\bar{x}_2^{'} = \bar{X}_2(1+e_2^{'})$, $s_{x_2}^2 = S_{x_2}^2 + e_3$, $s_{x_2}^{'2} = S_{x_2}^2 + e_3^{'}$ so that on ignoring fpc (finite population correction),

$$E(e_0) = E(e_1) = E(e_2) = E(e_3) = E(e_2^{'}) = E(e_3^{'}) = 0 \, , \; E(e_0^2) = \frac{\mu_{200}}{n\bar{Y}^2} \, ,$$

$$E(e_1^2) = \frac{\mu_{020}}{n\bar{X}_1^2} \, , \; E(e_2^2) = \frac{\mu_{002}}{n\bar{X}_2^2} \, , \; E(e_2^{'2}) = \frac{\mu_{002}}{n'\bar{X}_2^2} \, , \; E(e_2 e_2^{'}) = \frac{\mu_{002}}{n'\bar{X}_2^2} \, ,$$

$$E(e_3^2) = \frac{\mu_{002}^2(\beta_{2x_2} - 1)}{n} \, , \; E(e_3^{'2}) = \frac{\mu_{002}^2(\beta_{2x_2} - 1)}{n'} \, , \; E(e_3 e_3^{'}) = \frac{\mu_{002}^2(\beta_{2x_2} - 1)}{n'} \, ,$$

$$E(e_0 e_1) = \frac{\mu_{110}}{n\bar{Y}.\bar{X}_1} \, , \; E(e_0 e_2) = \frac{\mu_{101}}{n\bar{Y}.\bar{X}_2} \, , \; E(e_0 e_2^{'}) = \frac{\mu_{101}}{n'\bar{Y}.\bar{X}_2} \, , \; E(e_0 e_3) = \frac{\mu_{102}}{n\bar{Y}} \, ,$$

$$E(e_0 e_3^{'}) = \frac{\mu_{102}}{n'\bar{Y}} \, , \; E(e_1 e_2) = \frac{\mu_{011}}{n\bar{X}_1\bar{X}_2} \, , \; E(e_1 e_2^{'}) = \frac{\mu_{011}}{n'\bar{X}_1\bar{X}_2} \, , \; E(e_1 e_3) = \frac{\mu_{012}}{n\bar{X}_1} \, ,$$

$$E(e_1 e_3^{'}) = \frac{\mu_{012}}{n'\bar{X}_1} \, , \; E(e_2 e_3) = \frac{\mu_{003}}{n\bar{X}_2} \, , \; E(e_2 e_3^{'}) = \frac{\mu_{003}}{n'\bar{X}_2} \, , \; E(e_2^{'} e_3) = \frac{\mu_{003}}{n'\bar{X}_2} \, ,$$

$$E(e_2^{'} e_3^{'}) = \frac{\mu_{003}}{n'\bar{X}_2} \qquad (2.1)$$

where $\beta_{2x_2} = \mu_{004}/\mu_{002}^2$ is the coefficient of kurtosis of x_2 , $\gamma_{1X_2} = \sqrt{\beta_{1X_2}}$ and $\tau' = \left(\frac{1}{n} - \frac{1}{n'}\right)$. Further, it is assumed that the sample size is large enough to ignore terms of e_i 's greater than two, to justify first order of approximation [see Murthy (1967)]. Now expanding $\hat{R}_g = g(t)$ about the point $T = (R, \bar{X}_2, \bar{X}_2, S_{x_2}^2, S_{x_2}^2)$ in third order Taylor's series, we have

$$\hat{R}_g = g(T) + (\hat{R} - R)g_0 + (\bar{x}_2 - \bar{X}_2)g_1 + (\bar{x}_2^{'} - \bar{X}_2)g_2 + (s_{x_2}^2 - S_{x_2}^2)g_3$$
$$+ 1/2!\{(\hat{R} - R)^2 g_{00} + (\bar{x}_2 - \bar{X}_2)^2 g_{11} + (\bar{x}_2^{'} - \bar{X}_2)^2 g_{22} + (s_{x_2}^2 - S_{x_2}^2)^2 g_{33}$$

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$$
\begin{aligned}
&+\left(s_{x_2}^{'2}-S_{x_2}^{2}\right)^{2} g_{44}+\left(s_{x_2}^{'2}-S_{x_2}^{2}\right) g_{4}+2\left(\hat{R}-R\right)\left(\bar{x}_2-\bar{X}_2\right) g_{01}+2\left(\hat{R}-R\right)\left(\bar{x}_2^{'}-\bar{X}_2\right) g_{02} \\
&+2\left(\hat{R}-R\right)\left(s_{x_2}^{2}-S_{x_2}^{2}\right) g_{03}+2\left(\hat{R}-R\right)\left(s_{x_2}^{'2}-S_{x_2}^{2}\right) g_{04}+2\left(\bar{x}_2-\bar{X}_2\right)\left(\bar{x}_2^{'}-\bar{X}_2\right) g_{12} \\
&+2\left(\bar{x}_2-\bar{X}_2\right)\left(s_{x_2}^{2}-S_{x_2}^{2}\right) g_{13}+2\left(\bar{x}_2-\bar{X}_2\right)\left(s_{x_2}^{'2}-S_{x_2}^{2}\right) g_{14} \\
&+2\left(\bar{x}_2^{'}-\bar{X}_2\right)\left(s_{x_2}^{2}-S_{x_2}^{2}\right) g_{23}+2\left(\bar{x}_2^{'}-\bar{X}_2\right)\left(s_{x_2}^{'2}-S_{x_2}^{2}\right) g_{24}+2\left(s_{x_2}^{2}-S_{x_2}^{2}\right)\left(s_{x_2}^{'2}-S_{x_2}^{2}\right) g_{34}\Big\} \\
&+1/3!\left\{\left(\hat{R}-R\right)\frac{\partial}{\partial \hat{R}}+\left(\bar{x}_2-\bar{X}_2\right)\frac{\partial}{\partial \bar{x}_2}+\left(\bar{x}_2^{'}-\bar{X}_2\right)\frac{\partial}{\partial \bar{x}_2^{'}}+\left(s_{x_2}^{2}-S_{x_2}^{2}\right)\frac{\partial}{\partial s_{x_2}^{2}}+\left(s_{x_2}^{'2}-S_{x_2}^{2}\right)\frac{\partial}{\partial s_{x_2}^{'2}}\right\}^{3} g(t_*)
\end{aligned} \tag{2.2}
$$

where $t_* = (\hat{R}_*, \bar{x}_{2*}, \bar{x}_{2*}^{'}, s_{x_2*}^{2}, s_{x_2*}^{'2})$ such that $\hat{R}_* = R+\theta\left(\hat{R}-R\right)$,

$\bar{x}_{2*} = \bar{X}_2+\theta\left(\bar{x}_2-\bar{X}_2\right)$, $\bar{x}_{2*}^{'} = \bar{X}_2+\theta\left(\bar{x}_2^{'}-\bar{X}_2\right)$, $s_{x_2*}^{2} = S_{x_2}^{2}+\theta\left(s_{x_2}^{2}-S_{x_2}^{2}\right)$,
$s_{x_2*}^{'2} = S_{x_2}^{2}+\theta\left(s_{x_2}^{'2}-S_{x_2}^{2}\right)$ where $0<\theta<1$;

Taking expectation on both sides of (2.2) and, using regularity conditions (1.2) and results given in (2.1) upto the first degree of approximation, we have

$$E(\hat{R}_g)-R = Bias(\hat{R}_g)$$

$$
\begin{aligned}
&=\frac{R}{n}\left(\frac{\mu_{020}}{\bar{X}_1^{2}}-\frac{\mu_{110}}{\bar{Y}.\bar{X}_1}\right)+\frac{1}{2}\left\{\frac{\mu_{002}}{n} g_{11}+\frac{\mu_{002}}{n'}\left(g_{22}+g_{12}\right)+\frac{\mu_{002}^{2}}{n}\left(\beta_{2x_2}-1\right) g_{33}\right. \\
&+\frac{\mu_{002}^{2}}{n'}\left(\beta_{2x_2}-1\right) g_{44}+2\tau' R\bar{X}_2\left(\frac{\mu_{101}}{\bar{Y}.\bar{X}_2}-\frac{\mu_{011}}{\bar{X}_1.\bar{X}_2}\right) g_{01}+2\tau' R\left(\frac{\mu_{102}}{\bar{Y}}-\frac{\mu_{012}}{\bar{X}_1}\right) g_{03} \\
&\left.+2\frac{\bar{X}_2}{n}\left(\frac{\mu_{003}}{\bar{X}_2}\right) g_{13}+2\frac{\bar{X}_2}{n'}\left(\frac{\mu_{003}}{\bar{X}_2}\right)\left(g_{14}+g_{23}+g_{24}\right)+\frac{2}{n'}\mu_{002}^{2}\left(\beta_{2x_2}-1\right) g_{24}\right\}
\end{aligned} \tag{2.3}
$$

Square both sides of (2.2) and taking expectation, upto the first degree of approximation, we have

$$MSE(\hat{R}_g) = E(\hat{R}_g-R)^2$$

$$
\begin{aligned}
&= E\left\{\mathrm{R}^2\left(\mathrm{e}_0-e_1\right)^2+\bar{X}_2^2\left(e_2-e_2^{'}\right)^2 g_1^2+\left(e_3-e_3^{'}\right)^2 g_3^2+2R\bar{X}_2\left(e_0-e_1\right)\left(e_2-e_2^{'}\right) g_1\right. \\
&\left.+2\mathrm{R}\left(\mathrm{e}_0-e_1\right)\left(e_3-e_3^{'}\right) g_3+2\bar{X}\left(e_2-e_2^{'}\right)\left(e_3-e_3^{'}\right) g_1 g_3\right\}
\end{aligned}
$$

$$= \frac{1}{n} R^2 \left(\frac{\mu_{200}}{\bar{Y}^2} + \frac{\mu_{020}}{\bar{X}_1^2} - 2\frac{\mu_{110}}{\bar{Y}.\bar{X}_1} \right) + \tau' \left\{ \bar{X}_2^2 \left(\frac{\mu_{002}}{\bar{X}_2^2} \right) g_1^2 + \mu_{002}^2 \left(\beta_{2x_2} - 1 \right) g_3^2 \right.$$

$$\left. + 2\bar{X}_2 \frac{\mu_{003}}{\bar{X}_2} g_1 g_3 + 2R\bar{X}_2 \left(\frac{\mu_{101}}{\bar{Y}.\bar{X}_2} - \frac{\mu_{011}}{\bar{X}_1.\bar{X}_2} \right) g_1 + 2R \left(\frac{\mu_{102}}{\bar{Y}} - \frac{\mu_{012}}{\bar{X}_1} \right) g_3 \right\}$$

$$= MSE(\hat{R}) + \tau' \left\{ \begin{array}{l} \bar{X}_2^2 C_2^2 \left(g_1^2 + 2\frac{R}{\bar{X}_2} C g_1 \right) + \mu_{002}^2 \left(\beta_{2x_2} - 1 \right) g_3^2 \\ + 2R\mu_{002} \left(\lambda_{02} - \lambda_{12} \right) g_3 + 2\mu_{003} g_1 g_3 \end{array} \right\}$$

3. OPTIMUM AND ESTIMATED OPTIMUM VALUES

Partially differentiating (2.4) w r t g_1 and g_3, the optimum values of g_1 and g_3 minimizing the $MSE(\hat{R}_g)$ are

$$g_{1opt} = \frac{R\left\{ \mu_{003} \left(\lambda_{02} - \lambda_{12} \right) - \bar{X}_2 C_2^2 C \mu_{002} \left(\beta_{2x_2} - 1 \right) \right\}}{\left(\beta_{2x_2} - \beta_{1x_2} - 1 \right) \mu_{002}^2} = RB_1 (say) \quad \text{and}$$

$$g_{3opt} = \frac{R\left\{ \bar{X}_2 C_2^2 C \mu_{003} - C\mu_{002}^2 \left(\lambda_{02} - \lambda_{12} \right) \right\}}{\left(\beta_{2x_2} - \beta_{1x_2} - 1 \right) \mu_{002}^3} = RB_2 (say) \tag{3.1}$$

The minimum value of mean square error $MSE(\hat{R}_g)$ is given by

$$MSE(\hat{R}_g)_{\min} = MSE(\hat{R}) - \tau' \left[R^2 C_2^2 C^2 + \frac{R^2 \left\{ \left(\lambda_{02} - \lambda_{12} \right) - C_2 C \gamma_{1X_2} \right\}^2}{\left(\beta_{2x_2} - \beta_{1x_2} - 1 \right)} \right] \tag{3.2}$$

The optimum value of the parameters g_1 and g_3, may not be known in practice hence the alternative is to replace the parameters involved therein by their unbiased or consistent estimators which result in the following estimators

$$\hat{g}_{1opt} = \frac{\hat{R}\left\{ \hat{\mu}_{003} \left(\hat{\lambda}_{02} - \hat{\lambda}_{12} \right) - \bar{x}_2 \hat{C}_2^2 \hat{C} \hat{\mu}_{002} \left(\hat{\beta}_{2x_2} - 1 \right) \right\}}{\left(\hat{\beta}_{2x_2} - \hat{\beta}_{1x_2} - 1 \right) \hat{\mu}_{002}^2} = \hat{R}\hat{B}_1 (say)$$

$$\hat{g}_{3opt} = \frac{\hat{R}\left\{ \bar{x}_2 \hat{C}_2^2 \hat{C} \hat{\mu}_{003} - \hat{C}\hat{\mu}_{002}^2 \left(\hat{\lambda}_{02} - \hat{\lambda}_{12} \right) \right\}}{\left(\hat{\beta}_{2x_2} - \hat{\beta}_{1x_2} - 1 \right) \hat{\mu}_{002}^3} = \hat{R}\hat{B}_2 (say) \tag{3.3}$$

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$\hat{\mu}_{003} = m_{003}$, $\hat{\mu}_{002} = m_{002}$, $\hat{\lambda}_{02} = \frac{m_{102}}{\bar{y}m_{002}}$, $\hat{\lambda}_{12} = \frac{m_{012}}{\bar{x}_1 m_{002}}$, $\hat{C}_2^2 = \frac{m_{002}}{\bar{x}_2^2}$,

$\hat{C} = \frac{1}{\bar{x}_2 \hat{C}_2^2}\left(\frac{m_{101}}{\bar{y}} - \frac{m_{011}}{\bar{x}_1}\right)$, $\hat{\beta}_{2x_2} = \frac{m_{004}}{m_{002}^2}$, $\hat{\beta}_{1x_2} = \frac{m_{003}^2}{m_{002}^3}$,

$$\hat{B}_1 = \frac{\left\{\hat{\mu}_{003}\left(\hat{\lambda}_{02} - \hat{\lambda}_{12}\right) - \bar{x}_2 \hat{C}_2^2 \hat{C} \hat{\mu}_{002}\left(\hat{\beta}_{2x_2} - 1\right)\right\}}{\left(\hat{\beta}_{2x_2} - \hat{\beta}_{1x_2} - 1\right)\hat{\mu}_{002}^2} \text{ and}$$

$$\hat{B}_2 = \frac{\left\{\bar{x}_2 \hat{C}_2^2 \hat{C} \hat{\mu}_{003} - \hat{C}\hat{\mu}_{002}^2\left(\hat{\lambda}_{02} - \hat{\lambda}_{12}\right)\right\}}{\left(\hat{\beta}_{2x_2} - \hat{\beta}_{1x_2} - 1\right)\hat{\mu}_{002}^3}$$

The generalized double sampling estimator $\hat{R}_g$ attains the minimum mean square error in (3.2) if the conditions from (1.2), (3.1) and (3.2) are satisfied for the estimator $\hat{R}_g$. This means that the functions $\hat{R}_g = g(\hat{R}, \bar{x}_2, \bar{x}_2^{'}, s_{x_2}^2, s_{x_2}^{'2})$ as an estimator of R should not involve only $g(\hat{R}, \bar{x}_2, \bar{x}_2^{'}, s_{x_2}^2, s_{x_2}^{'2})$ but also g_{1opt} and g_{3opt} for the conditions (3.1) and (3.2) to be satisfied. Thus, we get the resulting estimator as a function $g(\hat{R}, \bar{x}_2, \bar{x}_2^{'}, s_{x_2}^2, s_{x_2}^{'2}, g_{1opt}, g_{3opt})$ satisfying the conditions (1.2) along with the conditions mentioned in (3.3) to attain the minimum mean square error in (3.2). Replacing the unknown parameters g_{1opt} and g_{3opt} in $g(\hat{R}, \bar{x}_2, \bar{x}_2^{'}, s_{x_2}^2, s_{x_2}^{'2}, g_{1opt}, g_{3opt})$, we get the estimator as a function $\hat{R}_g^* = g(\hat{R}, \bar{x}_2, \bar{x}_2^{'}, s_{x_2}^2, s_{x_2}^{'2}, \hat{g}_{1opy}, \hat{g}_{3opt})$ or equivalently the estimator as the function $g^*(\hat{R}, \bar{x}_2, \bar{x}_2^{'}, s_{x_2}^2, s_{x_2}^{'2}, \hat{B}_1, \hat{B}_2) = g^*(t^*)$ depending upon the estimated optimum values. Now expanding $g^*(t^*)$ about the point $T^* = (R, \bar{X}_2, \bar{X}_2, S_{x_2}^2, S_{x_2}^2, B_1, B_2)$ in the Taylor's series, we have

$$g^*(t^*) = g^*(T^*) + \left(\hat{R} - R\right)g_0^* + \left(\bar{x}_2 - \bar{X}_2\right)g_1^* + \left(\bar{x}_2^{'} - \bar{X}_2\right)g_2^* + \left(s_{x_2}^2 - S_{x_2}^2\right)g_3^*$$
$$+ \left(s_{x_2}^{'2} - S_{x_2}^2\right)g_4^* + \left(\hat{B}_1 - B_1\right)g_5^* + \left(\hat{B}_2 - B_2\right)g_6^* + \ldots \quad (3.4)$$

where $g^*(T^*) = R$ and $g_0^* = 1$

$$g^*(t^*) - R = \left(\hat{R} - R\right) + \left(\bar{x}_2 - \bar{X}_2\right)g_1^* + \left(\bar{x}_2^{'} - \bar{X}_2\right)g_2^* + \left(s_{x_2}^2 - S_{x_2}^2\right)g_3^*$$
$$+ \left(s_{x_2}^{'2} - S_{x_2}^2\right)g_4^* + \left(\hat{B}_1 - B_1\right)g_5^* + \left(\hat{B}_2 - B_2\right)g_6^* + \ldots \quad (3.5)$$

Squaring both the sides of (3.5) and taking expectation, we see that the MSE of $g^*(t^*)$, to the first degree of approximation, attains minimum value of MSE of $\hat{R}_g$ given by (3.2) if $g_5^* = g_6^* = 0$. Thus the estimator taken as function $\hat{R}_{ge} = g^*(t^*) = g^*(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, \hat{B}_1, \hat{B}_2)$ depending upon the estimated values attains $MSE(\hat{R}_g)_{\min}$ if

$$g^*(t^*)\big|_{T^*} = R,\ g_0^*\big|_{T^*} = 1,\ g_1^*\big|_{T^*} = -g_2^*\big|_{T^*},\ g_3^*\big|_{T^*} = -g_4^*\big|_{T^*},$$

$$g_{00}\big|_{T^*} = 0,\ g_{01}\big|_{T^*} = -g_{02}\big|_{T^*},\ g_{03}\big|_{T^*} = -g_{04}\big|_{T^*},$$

$$g_1^*\big|_{T^*} = g_{1opt},\ g_3^*\big|_{T^*} = g_{3opt},\ g_5^* = 0,\ g_6^* = 0 \qquad (3.6)$$

Satisfying the conditions in (3.6), some particular estimators depending on the estimated optimum values $\hat{g}_{1opt}$, $\hat{g}_{2opt}$ and attaining the minimum mean square error in (3.7), are given in the following section.

4. CONCLUDING REMARKS

(a) The optimum values g_{1opt} and g_{3opt} of g_1 and g_3 respectively minimize the mean square error of $\hat{R}_g = g(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2)$ and the resulting mean square error is given by

$$MSE(\hat{R}_g)_{\min} = MSE(\hat{R}) - \tau'\left[R^2C_2^2C^2 + \frac{R^2\left\{(\lambda_{02} - \lambda_{12}) - C_2C\gamma_{1X_2}\right\}^2}{\left(\beta_{2x_2} - \beta_{1x_2} - 1\right)}\right] \qquad (4.1)$$

Further the mean square error of $\hat{R}_{gd} = g(\hat{R}, \bar{x}_2, \bar{x}_2')$ satisfying the regularity conditions is given by

$$MSE(\hat{R}_{gd})_{\min} = MSE(\hat{R}) - \tau' R^2C_2^2C^2 \qquad (4.2)$$

From (4.1) and (4.2), we have

$$MSE(\hat{R}_g)_{\min} = MSE(\hat{R}_{gd})_{\min} - \tau'\left[\frac{R^2\left\{(\lambda_{02} - \lambda_{12}) - C_2C\gamma_{1X_2}\right\}^2}{\left(\beta_{2x_2} - \beta_{1x_2} - 1\right)}\right] \qquad (4.3)$$

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Showing that the class of estimators represented by $\hat{R}_g$ contains more efficient estimators than those in the class represented by $\hat{R}_{gd}$ in the sense of having lesser mean square error.

(b) From the class of estimators represented by $\hat{R}_g$, considering the estimators

$\hat{R}_3 = \hat{R} + k_1(\bar{x}_2 - \bar{x}_2^{'}) + k_2(s_{x_2}^2 - s_{x_2}^{'2})$ and $\hat{R}_4 = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}_2^{'}}\right)^{k_1}\left(\frac{s_{x_2}^2}{s_{x_2}^{'2}}\right)^{k_2}$, we find that $g_1 = k_1$ and $g_3 = k_2$ for $\hat{R}_3$, and $g_1 = k_1\frac{R}{\bar{X}_2}$ and $g_3 = k_2\frac{R}{S_{x_2}^2}$ for $\hat{R}_4$. By equating these values of g_1 and g_3 for $\hat{R}_3$ and $\hat{R}_4$ to g_{1opt} and g_{3opt} in (3.1), we observe that the estimator $\hat{R}_3$ for $k_1 = g_{1opt}$ and $k_2 = g_{3opt}$; and the estimator $\hat{R}_4$ for $k_1 = g_{1opt}\frac{\bar{X}_2}{R}$ and $k_2 = g_{3opt}\frac{S_{x_2}^2}{R}$, attain the minimum mean square error of $\hat{R}_{gd}$ given in (3.2) or (4.1).

(c) The optimum values of $k_1 = g_{1opt}$, $k_2 = g_{3opt}$ for $\hat{R}_3$ and $k_1 = g_{1opt}\frac{\bar{X}_2}{R}$, $k_2 = \frac{S_{x_2}^2}{R}g_{3opt}$ for $\hat{R}_4$ are rarely known in practice, hence replacing unknown parameters involved therein by their unbiased or consistent estimators, we get the estimated optimum values $\hat{k}_1 = \hat{g}_{1opt}$, $\hat{k}_2 = \hat{g}_{3opt}$ for $\hat{R}_3$ and $\hat{k}_1 = \hat{g}_{1opt}\frac{\bar{x}_2}{\hat{R}} = \bar{x}_2\hat{B}_1$, $\hat{k}_2 = \frac{s_{x_2}^2}{\hat{R}}\hat{g}_{3opt} = s_{x_2}^2\hat{B}_2$ for $\hat{R}_4$ so that the estimators depending upon the estimated optimum values corresponding to $\hat{R}_3$ and $\hat{R}_4$ become $\hat{R}_{3e} = \hat{R} + \hat{k}_1(\bar{x}_2 - \bar{x}_2^{'}) + \hat{k}_2(s_{x_2}^2 - s_{x_2}^{'2})$ and $\hat{R}_{4e} = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}_2^{'}}\right)^{\hat{k}_1}\left(\frac{s_{x_2}^2}{s_{x_2}^{'2}}\right)^{\hat{k}_2}$ which satisfy all the regularity conditions in (3.6) for the generalised estimator $\hat{R}_{ge}$ depending upon estimated optimum values, attain the minimum mean square error in (3.2) or (4.1).

(d) Similar results may be derived for the estimators developed for estimating the product P on the same lines of $\hat{R}_g$ and $\hat{R}_{ge}$.

(e) Single sampling results may be easily found by replacing n' by N .

5. EMPIRICAL STUDY

An empirical study was carried out on the population given in Sukhatme and Sukhatme (1997, pg 185). It consists of 34 villages in Lucknow subdivision (India) and for calculation purpose it has been assumed that n' = 15 and n = 4. The characteristics being

Y : Area under wheat 1937
X_1 : Total cultivated area in 1931
X_2 : Area under wheat 1936

where the following values were obtained

$\bar{Y} = 195.5294$ $\bar{X}_1 = 765.3530$ $\bar{X}_2 = 218.4118$

$S_y^2 = 22772.68$ $S_{x_1}^2 = 222931.4$ $S_{x_2}^2 = 28123.22$

$\rho_{01} = 0.9000253$ $\rho_{02} = 0.8312208$ $\rho_{12} = 0.8307546$

The minimum mean square error and the percent gain in efficiency w.r.t. $\hat{R}$ of $\hat{R}_{gd}$ and $\hat{R}_g$ are given in table 1.

Table 1: Minimum Mean Square Error and Percent Gain in Efficiency

Estimator	$\hat{R}$	$\hat{R}_{gd}$	$\hat{R}_g$
$MSE(.)_{min}$	0.00755	0.00735	0.00731
$Gain(\hat{R},.)$	0	2.62778	3.202742

The table 1 exhibit that there is considerable gain in efficiency while using the proposed class of estimators and therefore the proposed class of estimators can be preferred.

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