SUBDIRECTLY IRREDUCIBLE P NEAR-RINGS

D B Ratnakar[1], P.Vijaya Vani[2], B.Vijaya Kumar[3], P.Koteswara Rao[4]

Abstracts:In this paper, we study p-near-rings (p is a prime number). Ratiff [10] in his thesis studied p-near-rings. First we present some results about p-near-rings, like (i). Every p-near-ring with left identity is zero symmetric, (ii). Every p-near-ring has IFP. Next we study the following properties of p-near-rings like (i). Every distributive idempotent element is central, (ii). If p-near-ring N has a multiplicative identity, then all idempotents are central, (iii). Every non-trivial p-near-ring has a family of completely prime ideals with trivial intersection, (iv).If every non-trivial homomorphic image of p-near-ring N containing a non-zero central idempotent is commutative near-ring, (v). Every distributively generated p-near-ring with identity is a p-ring, (vi). Zero-symmetric subdirectly irrerducible IFP p-near-ring N which is not simple, then every non-zero ideal of N contains no non-zero idempotent, (vii). Every subdirectly irreducible p-near-ring with left identity is integral, (viii). Every subdirectly irreducible p-near-ring which is zero-symmetric with non-zero distributive elements fulfills cancellation laws, (N, +) is commutative or (N, ·) is commutative or N ∈ η_1 and 0, 1 are only idempotents, (ix). Every distributively generated p-near-ring is commutative ring and (x). A p-near-ring with weak commutative and non-zero distributive elements in every homomorphic image is isomorphic to a subdirect product of copies of the field Z_P hence p-near-ring [10].

Keywords: p-near-ring , distributively generated p-near-ring , subdirectly irrerducible IFP p-near-ring,

1.1 Definition : Suppose p is a prime number. A near-ring N is called a p-near-ring provided that for all x ∈ N : $x^P = x$ and px = 0.

1.2 Theorem [9]: Every p-near-ring with left identity is zero symmetric.

Proof: Suppose N is a p-near-ring.

Suppose e is a left identity and x ∈ N.

By mathematical induction

$(e + x0)^P = e + px0 = e + 0 = e$

$\therefore (e + x0)^P = e \quad \therefore \quad e + x0 = e$

$\Rightarrow \quad x0 = 0$

∴ p – near-ring N is zero-symmetric.

1.3 Theorem : If N is a p-near-ring then ab = 0 ⇒ ba = 0 and anb = 0 for all n ∈ N. i.e Every p-near-ring has IFP.

Proof: Since $a^P = a \, \forall \, a \in N$, N has no non-zero nilpotent elements i.e $a^k = 0 \Rightarrow a = 0$ for any positive integer k.

If a, b ∈ N and ab = 0 then ba = 0.

For, suppose ab = 0

$(ba)^2 = (ba)(ba) = b(ab)a = b0a = b0 = 0$

$\therefore (ba)^2 = 0 \Rightarrow ba = 0 \quad \therefore ab = 0 \Rightarrow ba = 0$

If $ab = 0 \Rightarrow anb = 0$ for every $n \in N$.

Suppose $ab = 0$, $n \in N$.

$$(anb)^2 = (anb)(anb) = an(ba)nb = an0nb = an0 = 0$$

$\therefore (anb)^2 = 0 \Rightarrow anb = 0. \quad \therefore ab = 0 \Rightarrow anb = 0.$

1.4 Theorem: Let N be a p-near-ring then we have

(a) Every distributive idempotent element is central.
(b) If for every idempotent element e and every element $x \in N$, $x^2e = (xe)^2$.
(c) If N has a multiplicative identity element then all idempotents are central.

Proof: (a). Suppose $e \in N$ is an idempotent and $x \in N$.
First we show that $ex = exe$

$(ex - exe)\,ex = (exe-exe)\,x = 0x = 0$

$\therefore (ex - exe)\,ex = ex\,(ex - exe) = 0$ (since $ab = 0 \Rightarrow ba = 0$)

$(ex - exe)\,e = exe - exe^2 = 0$

$\therefore (ex - exe)\,e = 0 \Rightarrow e\,(ex - exe) = 0$

Now $(ex - exe)^2 = (ex - exe)(ex - exe)$
$= ex\,(ex - exe) + (-exe)(ex - exe)$
$= 0 - ex0 = 0 - 0 = 0$

$\therefore (ex - exe)^2 = 0 \Rightarrow ex - exe = 0$

$\therefore ex = exe$

Suppose e is distributively idempotent element.

$\therefore e\,(xe - exe) = 0 \Rightarrow (xe - exe)e = 0$ (since $ab = 0 \Rightarrow ba = 0$)
$\Rightarrow xee - exee = 0 \Rightarrow xe - exe = 0$
$\therefore xe = exe \quad \therefore ex = xe.$

$\therefore$ Every distributive idempotent element is central.

(b) Suppose e is an idempotent element and x is any element in N.

$(x - xe)ex = 0 \Rightarrow ex(x-xe) = 0$ (since $ab = 0 \Rightarrow ba = 0$)
$\Rightarrow xex\,(x - xe) = 0 \Rightarrow xexe\,(x - xe) = 0$
(by IFP)
$\Rightarrow xe(x-xe)xe = 0$

Consider $((x - xe)xe)^2 = (x-xe)xe(x-xe)xe = (x-xe)0 = 0$

$\therefore ((x - xe)xe)^2 = 0 \Rightarrow ((x - xe)xe = 0$

$\Rightarrow xxe - xexe = 0 \Rightarrow x^2e - (xe)^2 = 0 \Rightarrow x^2e = (xe)^2$

$\therefore x^2e = (xe)^2.$

(c) Suppose N has multiplicative identity 1 and e is an idempotent.

$(e - 1)\, e = 0 \quad \Rightarrow \quad e\,(e-1) = 0 \quad$ (since $ab = 0 \Rightarrow ba = 0$)

$(e - 1)\, xe = exe - xe$

Now $(exe - xe)^2 = (exe - xe)\,(exe - xe) = (exe - xe)\, e\,(e-1)\, xe$

$= (exe - xe)0xe = 0$

$\therefore (exe - xe)^2 = 0$

$\Rightarrow exe - xe = 0 \Rightarrow xe = exe$

But e is idempotent, we have $ex = exe$. $\therefore$ $ex = xe$

$\therefore$ All idempotent elements are central.

1.5 Theorem : A non-trivial p-near-ring N contains a family of completely prime ideals with trivial intersection.

Proof: Since N has no non-zero nilpotent elements, then N has multiplicative sub semi-groups which do not contain zero element. By Zorn's lemma, let M be any maximal multiplicative sub semi-group which do not contain zero element.

Define $A(M) = \{x \in N/\ xa = 0$ for atleast $a \in N\}$

Claim : A(M) is a prime ideal.

First we show that A(M) is normal subgroup of (N, +).

Let $u, v \in A(M)$

$\Rightarrow \exists\, a, b \in M \ni ua = 0, vb = 0 \quad \Rightarrow uab = 0, vab = 0$ (by IFP)

$\Rightarrow (u - v)ab = 0 \quad \Rightarrow (u - v) \in A(M)$

Let $u \in A(M)$ and $x \in N$ then since $u \in A(M) \Rightarrow \exists\, a \in M \ni ua = 0$

$\Rightarrow (x + u - x)a = xa + ua - xa = 0 \quad$ (since $ua = 0$)

$\Rightarrow (x + u - x) \in A(M)$

$\therefore$ A(M) is a normal subgroup of (N, +).

Let $x \in N$ and $u \in A(M)$

Since $u \in A(M)\ \exists\, a \in M \ni ua = 0$

$\Rightarrow uxa = 0 \quad$ (by IFP) $\quad \Rightarrow ux \in A(M)$

Let $x, y \in N$ and $u \in A(M)$

Since $u \in A(M)\ \exists\, a \in M \ni ua = 0$

Consider $[y(x+u) - yx]\, a = y\,(x+u)\, a - yxa = y(xa + ua) - yxa$

$= yxa - yxa = 0$

$\therefore [y(x+u) - yx] \in A(M) \quad \therefore$ A(M) is an ideal.

Suppose $x \notin M$. Then the multiplicative sub semi-group generated by M and x must contain zero. Since M has no non-zero nilpotent elements, some finite product containing x has atleast one factor and having atleast one factor from M must be zero. Repeated application of IFP, there exist an $m \in M$ such that xm is nilpotent so $xm = 0$

$\Rightarrow$ the set theoretical compliment of A(M) is M.

$\therefore$ A(M) is a completely prime ideal.

And clearly every non-zero element of M is excluded from atleast one of the prime ideals of A(M).

$\therefore$ N contains a family of completely prime ideals with trivial intersection.

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1.6 **Theorem :** N is a non-trivial p-near-ring with identity and every non trivial homomorphic image of N contains a non-zero central idempotent, then the additive group of N is commutative.

1.7 **Theorem :** Distributively generated p-near-ring N with identity is a commutative ring.

Proof: Suppose a is a distributive idempotent element in N by the previous 1.4 theorem(a), a is central. So by above 1.6 theorem , (N, +) is commutative. By a known result we know that (N, +) is a commutative.

By a Jacobson theorem for rings in [29]. $\therefore$ N is a p- ring.

1.8 **Theorem:** Every p-near-ring N with identity is a p-ring.

Proof: Suppose N is a p-near-ring with identity 1.

$\Rightarrow \bar{1}$ is non-zero central idempotent.

Any near-ring in any ring $N = \overline{N}/P$ where P = A(M).

So by 1.6 theorem (N, +) is commutative.

For every $a \in N$, a^{P-1} is an idempotent.

Since N has multiplicative identity 1, then all idempotents are central. Since every element a in N can be written as sum of idempotents. So N is distributively generated p-near-ring.

$\therefore$ (N, $\cdot$) is commutative. $\therefore$ N is a p-ring.

1.9 **Definition:** A near-ring N is called strongly uniform if $\forall n \in N$: (0: n) = 0 or (0 : n) = N but $\exists m \in N$: (0, m) = {0}.

1.10 **Note:** Suppose N is subdirectly irreducible zero-symmetric simple p-near-ring then N is strongly uniform or for all $x, y \in N$: xy = 0.

Proof: Since (0 : n) is an ideal in N. since N is subdirectly irreducible with IFP. $\Rightarrow$ (0 : n) = N or (0: n) = 0. $\therefore$ N is strongly uniform.

Suppose (0 : n) = N $\forall n \in N$.

Let $y \in N$ $\Rightarrow$ (0 : y) = N $\Rightarrow$ xy = 0 $\forall x \in N$. $\therefore$ $\forall x, y \in N$; xy = 0.

Main Theorems

2. 1 **Theorem**: Suppose N is zero-symmetric sub-directly irreducible IFP p-near-ring which is not simple, then no non-zero ideal contains a non-zero idempotent.

Proof: Suppose P is a non-zero ideal $\neq$ N . Suppose P contains a non-zero idempotent e $\neq 0$.

Claim: e is right identity.

Suppose e is not right identity. $\Rightarrow \exists\ x \in N \ni xe \neq x$

$\Rightarrow xe - x \neq 0$ $\Rightarrow xe - x \in (0 : e)$, $e \in (0 : e)$ and $e = e^2 = 0$, it is a contradiction. $\Rightarrow P = N$, it is a contradiction.

$\therefore$ P does not contain idempotent element .

$\therefore$ No non-zero ideal of N contains a non-zero idempotent.

2.2 **Theorem:** Suppose N is subdirectly irreducible p-near-ring which is not zero-symmetric nor constant then every non-zero ideal P which contains non-zero idempotent then $P = N_0$.

Proof: Suppose N is subdirectly irreducible p-near-ring which is not zero-symmetric nor constant.

Suppose P is a non-zero ideal. Clearly $P \subseteq (0: 0)$, the p has a non-zero idempotent $e \neq 0$ (by above 2.11 theorem). Then e is right identity in N.

Let $x \in (0 : 0) \Rightarrow x0 = 0$

$\therefore x = xe \in P$ (since e is a right identity and $e \in P$ & $\therefore xe \in P$).

$\therefore P = (0 : 0) = N_0$.

2.3 **Definition.** Consider the following properties :

(P_0) : $\forall\ x \in N\ \exists\ n(x) > 1 : x^{n(x)} = x$.

(P_1) : (P_0) and $N \in \eta_0$.

(P_2) : $\forall\ x, y \in N\ \exists\ n(x, y) > 1 : (xy - yx)^{n(x, y)} = xy - yx$ and $N \in \eta_0$.

(P_3) : $\forall\ x, y \in N : xyz = xzy$ (" Weak commutative").

(P_4) : $\forall\ x, y \in N\ \forall\ I \trianglelefteq N : xy \in I \Rightarrow yx \in I$.

2.4 **Theorem .** Every subdirectly irreducible p-near-ring N with left identity. Then n is integral (i.e has no non-zero zero divisors).

Proof: Suppose N is a subdirectly irreducible p-near-ring with left identity.

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Since N has left identity , then N is zero symmetric.

Since $a^p = a \; \forall \; a \in N$, N has no non-zero nilpotent elements i.e $a^k = 0$ for some $k \in N \Rightarrow a = 0$.

Suppose ab = 0 for some $a, b \in N$.

Then $(ba)^2 = (ba)(ba) = b(ab)a = b(0)a = b0 = 0$.

$\therefore (ba)^2 = 0 \Rightarrow ba = 0. \qquad \therefore ab = 0 \Rightarrow ba = 0.$

Suppose ab = 0 & $n \in N$.

Consider $(anb)^2 = (anb)(anb) = an(ba)\,nb = an\,(0)\,nb = an(0nb)$

$= an0 \qquad = 0.$

$\therefore (anb)^2 = 0 \Rightarrow anb = 0 \qquad \therefore ab = 0 \Rightarrow anb = 0 \; \forall \; n \in N.$

Let $x \in N^*$ Then $(\{x^k \,/\, k = 1,2,..p-1\}, \cdot)$ does not contain 0 and contained in the semi group M_x maximal for not containing 0 (by Zorn's lemma).

Let $I_x = \bigcup_{m \in M_x} (0 : m)$.

Consider (0 : m), (0 : n) for m , $n \in M_x$.

Clearly $mn \in M_x$.

Let $d \in (0 : m) \Rightarrow dm = 0 \qquad \Rightarrow dmn = 0$

$\Rightarrow d \in (0 : mn)$

$\therefore (0 : m) \subseteq (0 : mn)$

Let $d \in (0 : n) \Rightarrow dn = 0$

$\Rightarrow dmn = 0$ (by IFP of N) $\Rightarrow d \in (0 : mn)$

$\therefore (0 : n) \subseteq (0 : mn)$

$\therefore I_x$ is a union of directed set of ideals. $\therefore I_x$ is an ideal.

Claim: $x \notin I_x$

Suppose $x \in I_x \Rightarrow x \in (0 : m)$ for some $m \in M_x \Rightarrow xm = 0$

Since $x \in M_x$, $m \in M_x \Rightarrow xm \in M_x \Rightarrow 0 \notin M_x$.

It is a contradiction. $\therefore x \notin I_x$.

$\therefore \bigcap_{z \in N^*} I_z = \{0\}$.

Since N is subdirectly irreducible , $\exists\, y \in N^*$, $I_y = \{0\}$

Claim: $M_y = N^*$ Suppose $n \in N$, $n \notin M_y$

$\Rightarrow$ the sub semigroup generated by M_y and N contains 0. Such product has one of the following forms .

$m_1nm_2 = 0$, $nm^l=0$, $m^{ll}n=0$, $n=0$. (for $m_1, m_2, m^l, m^{ll}n \in M_y$.)

Suppose $m_1nm_2 = 0$

$\Rightarrow m_1n \in (0 : m_2)$ $\Rightarrow m_1n = 0$ since $(0 : m_2) \subseteq I_y = \{0\}$

$\Rightarrow m_1n = 0$ $\Rightarrow nm_1 = 0$ (by IFP of N)

$\Rightarrow n \in (0 : m_1) = \{0\}$ $\Rightarrow n = 0$

From the three cases, $n = 0$ $\therefore n = 0$

$M_y = N^*$ Let $x, y \in N$ & $x \neq 0$, $y \neq 0$

$x \neq 0$, $y \neq 0$ $\Rightarrow x, y \in N^*$ $\Rightarrow xy \in M_y$ $\Rightarrow xy \neq 0$

$\therefore$ N is integral.

2.5 **Theorem :** N is a subdirectly irreducible, zero-symmetric, p-near-ring with $N_d \neq \{0\}$, then N fulfils both cancellation laws, (N, +) is abelian and either $(N, \cdot)$ is commutative or $N \in \eta_1$ and 0 ,1 are only idempotents.

Proof: Suppose e is any non-zero idempotent and $d \in N_d$ and $d \neq 0$.

Let $n \in N$. $(ne - n)e = 0 \Rightarrow ne - n = 0$ (since $e \neq 0$ & N is integral)

$d(en - n) = den - dn = dn - dn = 0$.

(since every non-zero idempotent is right identity)

$\therefore en - n = 0$ (since $d \neq 0$) $\Rightarrow en = n$

$\therefore$ N has an identity 1 and each non-zero idempotent = 2.

Cancellation laws : suppose a, b , c $\in$ N with ab = ac , a $\neq$ 0.

If a is central, ab = ac

$\Rightarrow$ ba – ca = 0 $\Rightarrow$ (b – c)a = 0 $\Rightarrow$ b – c = 0 since a $\neq$ 0.

$\therefore$ b = c.

Suppose a is not central.

$\Rightarrow \exists$ f $\in$ N* $\ni$ af $\neq$ fa i.e af – fa $\neq$ 0

$\Rightarrow$ (af – fa) a $\neq$ 0

$(afa - faa)^p = (afa - faa)$

$(afa - faa)^{p-1} = e = 1$ (e $\neq$ 0, $e^2 = e$)

$\therefore (afa - faa)^{p-2} (afa - faa) a = 1$ $\therefore$ a has left inverse.

$\therefore$ b = c $\therefore$ ab = ac or ba = ca $\Rightarrow$ b = c. (N , +) is abelian.

2 = 1 + 1 Suppose 2 = 0 $\Rightarrow$ 2x = 0 $\forall$ x $\in$ N

$\Rightarrow$ (N , +) is abelian.

Suppose 2 $\neq$ 0. Then 2 is central or not central.

Suppose 2 is central

2 (n + m) = (1 + 1) n + m = n + m + n + m

(n + m) 2 = n2 + m2 = 2n + 2m

= n + n + m + m.

$\therefore$ n + m + n + M = n + n + m + m. $\Rightarrow$ m + n = n + m

$\therefore$ (N , +) is abelian.

Suppose 2 is not central. Since 2 $\neq$ 0, so 2 has left inverse.

Suppose u is left inverse of 2.

Consider (1 – 2u) 2 = 2 – 2u2 = 2 – 2·1 = 2 – 2 = 0

$\Rightarrow$ 1 – 2u = 0 since 2 $\neq$ 0

$\therefore$ 2u = 1 $\therefore$ u2 = 2u = 1 $\therefore$ u is right inverse of 2

Let $r \in N$ and put h = ur

h + h = ur + ur = (u + u) r = 2ur = 1r = r $\therefore$ h + h = r

Suppose $r = h^l + h^l \Rightarrow h + h = h^l + h^l$

$\Rightarrow 2h = 2h^l$ $\Rightarrow$ h = h^l

Suppose $n \in N$ and n (-1) = n

If n is central, n (-1) = n $\Rightarrow$ (-1)n = n

$\Rightarrow$ - n = n $\Rightarrow$ n + n = 0 $\Rightarrow$ 2n = 0 $\Rightarrow$ n = 0

Suppose n is not central.

Claim: n = 0 Suppose $n \neq 0 \Rightarrow$ n has left inverse $n^{-2.}$

Since n (-1) = n $\Rightarrow n^{-1}n(-1) = n^{-1}n \Rightarrow$ 1(-1) = 1 $\Rightarrow$ -1 = 1

2 = 1 + 1 = 1 + (-1) = 0. It is a contradiction.

$\therefore$ By a proposition (2.109(c) [21]) implies , (N , +) is abelian.

2.6 **Theorem** : Every distributively generated near-ring N is commutative.

Proof: Suppose $N \in \eta$; then $N_d \neq 0$. Since is N_i integral. (N , +) is abelian.

Since N_i is distributively generated abelian, so N_i is a ring (6.6c [1]) and N_i is commutative ring. Suppose $N_i \notin N$, so N_i is commutative ring.

Since all N_i 's are commutative near-rings, so N is a commutative ring.

2.7 **Corollary :** N is a p-near-ring with IFP and every non-zero homomorphic image of N contains non-zero distributive elements, then $N \in \eta_0$ and a subdirect product of near-fields.

Proof: Since N is a p-near-ring with IFP

$\therefore N_0 = (0 : 0) = N$ $\therefore N / N_0$ is constant. $\therefore$ N has a strong IFP.

Since $N \in \eta_0$ is p –ring , N is isomorphic to a subdirect product of simple integral near-ring $\in \eta_0$ with a right identity (9.10 [1]).

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$\Rightarrow$ Every N_i is abelian and either commutative or $N_i \in \eta_2$. So in any case N_i, is near-field.

2.8 Corollary : Every distributively generated p-near-ring is a subdirect product of commutative fields and hence a commutative ring.

2.9 Corollary: A p-near-ring with IFP and weak commutative is a subdirect product of p-near-ring $N_i \neq 0$.

Proof: Every $N_i \neq 0$ and is simple. Suppose N_i has one right identity $\forall$ x $\in N_i^*$, $x^{p-1} = e$. Every non-zero idempotent is a right identity $\forall$ x, y $\in N_i^*$: $xy = x^p y = x^{p-1}xy = exy = eyx = y^{p-1}yx = y^p x = yx$

$\therefore xy = yx$ $\therefore N_i$ is commutative,

$\therefore$ e is an identity & N_i is a simple integral domain, so a field.

2.10 Corollary : Suppose N is a p-near-ring with weak commutative and every non-zero homomorphic image has non-zero distributive elements. Then N is a subdirect product of commutative fields and hence commutative ring.

Proof: Since N is p-near-ring, so N has IFP $\Rightarrow N \in \eta_0$.

N is a subdirect product of subdirectly irreducible p-near-ring N_i.

Since $N_i \neq \{0\}$, so every non-zero idempotent of N_i is 2. Clearly $N_i \in \eta_0$

and by 2.16 theorem N_i is simple and N_i is commutative field.

$\therefore (N_i)_0 = N_i$, $(N_i)_d \neq \{0\}$,

$\therefore$ By above corollary, N_i is commutative field.

$\therefore$ N is a commutative ring with identity 2.

3. REFERENCES

1 Blackett D.V; "Simple and semisimple near-rings ",Proc. Amer. Math. Soc. 4(1953),772-785.

2 Frohlich A ; "Distributively generated near-rings, (I.Ideal Theory) ' , Proc. London Math. Soc. (3), 8(1958). 76-91.

3 Howard E.Bell; "Near-rings in which each element is a power of itself", Bul. Austral.Math.Soc.Vol.2(1970),363-368.

4 James R.Clay "The near-rings on groups of low order", Math Z. 104 (1968) ,364-371.

5 Ligh Steve ; "On Boolean near-rings" , Bull.Austral. Math. Soc. 1(1969), 375 - 379.

6 Ligh Steve; "On division near-rings", Cand.J.Math(to appear).

7 Pierce R.S' "Modules over commutative regular rings" , Mem. Amer. math.Soc.70.Providence Rhode Island, 1967.

8 Pilz.G ; "Near-rings, Theory and its Applications", North Holland 1983.

9 Plasser; "Subdirekte Darstellung Von Ringen und Fastringen Mit Boolschen Eigenschaften", Diplomarbeit, Univ, Linz, Austria,1971.

10 Ratliff, Ernest F; "Some results on p-near-rings and related near-rings", Ph.D., Dissertation, University of Oklahoma,1971.

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[1]Dr.D.Bala Ratnakar
Visiting Professor, SPAV (Estd by MHRD Govt of India)
Vijayawada, Andhra Pradesh, India.
[2]Dr.P.Vijaya Vani
Visiting Professor, Dept of IBS
Acharya Nagarajuna University , Guntur, Andhra Pradesh, India.
[3]Dr.B.Vijaya Kumar
Head & Professor, Dept of Maths
Andhar Christian College, Guntur, Andhra Pradesh, India.
[4]Prof.Dr.P.Koteswara Rao
Professor of Mathematics, Dept of Commerce & Business Admin.
Acharya Nagarajuna University , Guntur, Andhra Pradesh, India.