A NOTE ON THE INVENTORY MODEL FOR DETERIORATING ITEMS WITH TRAPEZOIDAL TYPE DEMAND RATE AND VARIABLE DETERIORATION

Trailokyanath Singh[1] Sudhir Kumar Sahu[2]

Abstract: Preparation of Papers - In this paper, an inventory model is developed for deteriorating items with trapezoidal type demand rate, that is, the demand rate is a piecewise linearly functions. The deterioration is assumed to be a variable deterioration rate. An inventory replenishment policy for such type of inventory model is considered. The numerical examples and sensitivity analysis are provided to illustrate the inventory model.

Keywords:Deteriorating items, inventory, trapezoidal type demand rate, variable deterioration, replenishment.

1. INTRODUCTION

Deterioration can not be avoided in business scenarios. Ghare and Schrader [1] categorized the inventory deterioration into three types: direct spoilage, physical depletion and deterioration. Direct spoilage refers to the unusable state of inventory items caused by breakage during transaction or by sudden accidental events. Commodities such as vegetables, fruits, food stuffs, etc. suffer from depletion by direct spoilage while kept in store. The quality and effectiveness of some medicines might be reduced in the event of non-functioning of refrigerator caused by sudden load shedding or absence of power supply for hours of together. Physical depletion refers to the highly volatile liquids such as alcohol, gasoline, turpentine etc., undergo physical depletion with time through the process of evaporation. Deterioration, on the other hand, refers to the slow but the gradual loss of qualitative properties of an item with the passage of time. For example electronics goods, radioactive substances, photographic films, food grain, etc., deteriorate through the gradual loss of potential or utility with time. In fact, no inventory item can avoid this kind of deterioration. This is inevitable.

The earliest work is due to Ghare and Schrader [1] who developed a simple EOQ (Economic Order Quantity) model with constant rate of deterioration. Covert and Philip [2] extended their model for variable rate of deterioration assuming a two parameter Weibull distribution. Misra [3] first developed the product lot size model in which both constant and variable rates of deterioration are considered.

In the classical EOQ (Economic Order Quantity) model developed in 1915, the demand rate of an item was assumed as constant. Thereafter, many inventory modelers developed the inventory literature considering the demand as constant. However, in the real market situations, the demand rate of any product is always in a dynamic state. Silver and Meal [4] first suggested a simple modification of the EOQ for the case of a varying demand. However, Donaldson [5], Silver [6], Ritchie [7-8] etc., made valuable contributions in this direction. For the review literature, it is found that most of the researchers have paid their attention only two types of time dependent demands, namely linear and exponential. A linearly time-varying demand indicates a uniform change in demand rate of the item per unit time which seldom occurs in the real market where as exponential time-varying demand indicates very rapid change in demand rate which is also unrealistic because the demand of any product can not undergo a rate which is as high as exponential demand is. Recently, Hill [9] first developed an inventory model with ramp type demand rate which is very commonly seen when some fresh fruit come to the market. In case of ramp type demand rate, the demand increases linearly at the beginning and then the market grows into a stable stage such that the demand becomes a constant until the end of the inventory cycle. He developed the inventory models for increasing demands followed by a constant demand derived the exact solution to compare with the Silver-Meal heuristic. Mandal and Pal [10] extended the inventory model with ramp type demand for the deteriorating items and allowing the shortage. Wu and Ouyang [11] extended the inventory model to include two different replenishment policies; (a) models starting with no shortages and (b) models starting with shortage. Deng Peter et al. [12] pointed out some questionable results of Mandal and Pal [10] and Wu and Ouyang [11] and then resolved the similar problem by offering a rigorous and efficient method to derive the optimal solution. Furthermore, Wu [13] investigated the inventory model with ramp type demand rate taking the rate of deterioration as the Weibull distribution. Skouri et al. [14] considered inventory models with ramp type demand rate, partial backlogging and Weibull deterioration rate. Giri, Jalan and Chaudhuri [15] extended the ramp type demand inventory model with a more generalized Weibull distribution deterioration. For more literatures of inventory models for deteriorating items, researchers can refer the performance works by Wu et al. [16], Manna and Chaudhuri [17], Benkkherouf [18], Goswami and Chaudhuri [19], Panda, Senapati and Basu [20], Giri et al [21], Wu [13] and Cheng and Wang [22].

In this paper, we consider inventory model for deteriorating items with trapezoidal type demand rate and variable deterioration. Assumption of the inventory system consists of several replenishments and all the ordering cycles are of fixed length on which only one of the ordering cycles is taken. Such type of demand pattern is generally seen in the case of any fad or seasonal goods coming to the market, e. g., mango, orange and some sea fish like hils herring. The demand rate for such items increases with time at the beginning of a season attains a peak and becomes steady at the middle of the season and finally decreases with time towards the end of the season. This type of demand behavior can be observed in some seasonal products in general. Such model was first studied by Cheng and Wang [22]. This model can be used for seasonal items like fruits, vegetables etc. whose deterioration rate increase with time. Thus the model considered above is suitable for items having variable deterioration rate, earlier models have considered items constant rate of deterioration. We think that such type of demand rate and deterioration is quite realistic and can provide a solid foundation for the future study of this kind of important inventory models with trapezoidal type demand rate and variable deterioration rate. A variable deterioration rate is used for the deteriorating items in order to make the model more generalized and realistic.

The rest of the paper is organized as follows. In the following section, we describe the notations and assumptions used throughout this paper. In section 3, we establish the mathematical model with shortage in inventory and the necessary conditions to find an optimal solution. In section 4, we use some numerical examples to illustrate the solution procedure. In section 5, sensitivity analysis is carried out. Finally, we make a summary of the paper.

2. NOTATIONS AND ASSUMPTIONS

In this paper, we extend Hill's ramp type demand rate to trapezoidal type demand rate. The fundamental assumption and notation used in this paper are given as below.

1. The replenishment is infinite, thus, replenishment is instantaneous.

2. The demand rate, $D(t)$, which is positive and consecutive, is assumed to be a trapezoidal type function of time, that is,

$$D(t) = \begin{cases} a_1 + b_1 t\,, & t \le \mu_1\,, \\ D_0\,, & \mu_1 \le t \le \mu_2\,, \\ a_2 - b_2 t\,, & \mu_2 \le t \le T \le \frac{a_2}{b_2}\,, \end{cases}$$

where μ_1 is the time point changing from the increasing linearly demand to constant demand, and μ_2 is the time point changing from constant demand to the decreasing linearly demand. (see Fig. 1.)

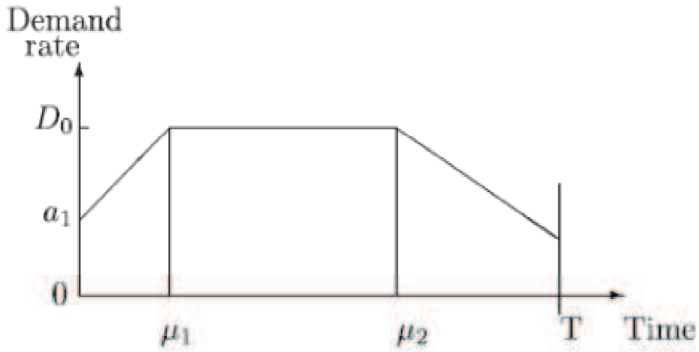

Fig. 1. A trapezoidal type function of the demand rate.

1. $I(t)$ is the level of inventory at time t , $0 \le t \le T$.

2. T is the fixed length of each ordering cycle.

3. θt is the variable deteriorating rate, $0 < \theta < 1$.

4. t_1 is the time when the inventory level reaches zero.

5. $t_1{}^*$ is the optimal point.

6. A_0 is the fixed ordering cost per order.

7. c_1 is the cost of each deteriorated item.

8. c_2 is the inventory holding cost per unit per unit time of time.

9. c_3 is the shortage cost per unit per unit of time.

10. S is the maximum inventory level for the ordering cycle such that $S = I(0)$.

11. Q is the ordering quantity per cycle.

12. $C_1(t_1)$ is the average total cost cost per unit time under the condition $t_1 \leq \mu_1$.

13. $C_2(t_1)$ is the average total cost cost per unit time under the condition $\mu_1 \leq t_1 \leq \mu_2$.

14. $C_3(t_1)$ is the average total cost cost per unit time under the condition $\mu_2 \leq t_1 < T$.

3. MATHEMATICAL FORMULATION

We consider the deteriorating inventory model with trapezoidal type demand rate. Replenishment occurs at time $t = 0$ when the inventory level attains its maximum. From $t = 0$ to t_1 , the inventory level reduces due to demand and deterioration. At t_1 , the inventory level achieves zero, then shortage is allowed to occur during the time interval (t_1, T) , and all of the demand during the shortage period (t_1, T) is completely backlogged. The total number of backlogged items is replaced by the next replenishment. According to the notations and assumptions mentioned above, the behavior of the inventory system at any time can be described by the following differential equations:

$$\frac{dI(t)}{dt} = -\theta t I(t) - D(t), \qquad 0 < t < t_1 \tag{1}$$

and

$$\frac{dI(t)}{dt} = -D(t), \qquad t_1 < t < T. \tag{2}$$

with boundary condition $I(t) = 0$.

In follows, we consider three possible cases based on the values of t_1 , μ_1 and μ_2 . These three cases are shown as follows. (see Fig. 2.).

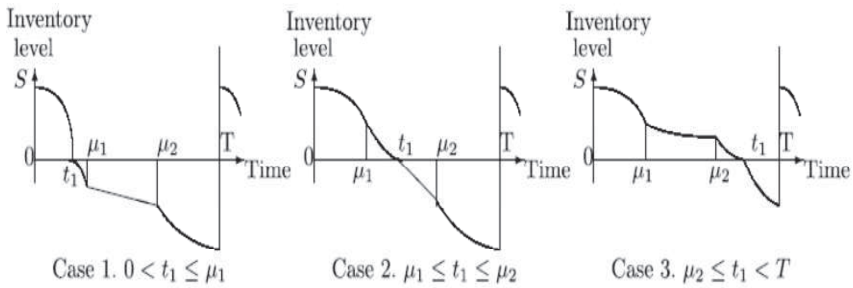

Fig. 2. Graphical representation of inventory level over the cycle.

Case 1. $0 < t_1 \le \mu_1$.

Due to reasons of deteriorating items and trapezoidal type market demand, the inventory level gradually diminishes during the period $[0, t_1]$ and ultimately falls to zero at time t_1. Then, from Eq. (1), we have

$$\frac{dI(t)}{dt} = -\theta t I(t) - (a_1 + b_1 t),\ \ 0 < t < t_1 \tag{3}$$

$$\frac{dI(t)}{dt} = -(a_1 + b_1 t)\ , \qquad t_1 < t < \mu_1\ , \tag{4}$$

$$\frac{dI(t)}{dt} = -D_0\ , \qquad \mu_1 < t < \mu_2, \tag{5}$$

and

$$\frac{dI(t)}{dt} = -(a_2 - b_2 t)\ , \qquad \mu_2 < t < T\ . \tag{6}$$

Solving the differential equations (3)-(6) with $I(t_1) = 0$ and neglecting the higher power of θ as $0 < \theta < 1$, we have

$$I(t) = \begin{bmatrix} \left\{a_1\left(t_1 + \frac{\theta}{6}{t_1}^3\right) + b_1\left(\frac{{t_1}^2}{2} + \frac{\theta}{8}{t_1}^4\right)\right\} \\ -\left\{a_1\left(t + \frac{\theta}{6}t^3\right) + b_1\left(\frac{t^2}{2} + \frac{\theta}{8}t^4\right)\right\} \end{bmatrix} e^{-\frac{\theta t^2}{2}}, \qquad . \qquad 0 \le t \le t_1, \tag{7}$$

$$I(t) = a_1(t_1 - t) + \frac{b_1}{2}({t_1}^2 - t^2), \quad t_1 \le t \le \mu_1 \tag{8}$$

$$I(t) = a_1 t_1 - D_0 t + \frac{b_1}{2}({t_1}^2 + {\mu_1}^2), \quad \mu_1 \le t \le \mu_2 \tag{9}$$

$$I(t) = a_1 t_1 - a_2 t + \frac{b_2}{2}(t^2 + \mu_2{}^2) + \frac{b_1}{2}(t_1{}^2 + \mu_1{}^2)\ , \qquad \mu_2 \le t \le T\ . \tag{10}$$

The beginning inventory level can be computed as

$$S = I(0) = a_1\left(t_1 + \frac{\theta}{6}t_1{}^3\right) + b_1\left(\frac{t_1{}^2}{2} + \frac{\theta}{8}t_1{}^4\right). \tag{11}$$

The total number of items which perish in the interval $[0, t_1]$, say D_T ,is

$$D_T = S - \int_0^{t_1} D(t)\, dt$$

$$= S - \int_0^{t_1}(a_1 + b_1 t)\, dt = \frac{\theta t_1^3}{2}\left(\frac{a_1}{3} + \frac{b_1}{4}t_1\right). \tag{12}$$

The total number of inventory carried during the interval $[0, t_1]$, say H_T , is

$$H_T = \int_0^{t_1} I(t)\, dt$$

$$= \int_0^{t_1}\left[\begin{bmatrix}\left\{a_1\left(t_1 + \frac{\theta}{6}t_1{}^3\right) + b_1\left(\frac{t_1{}^2}{2} + \frac{\theta}{8}t_1{}^4\right)\right\} \\ -\left\{a_1\left(t + \frac{\theta}{6}t^3\right) + b_1\left(\frac{t^2}{2} + \frac{\theta}{8}t^4\right)\right\}\end{bmatrix} e^{-\frac{\theta t^2}{2}}\right] dt$$

$$= a_1\left(\frac{t_1{}^2}{2} + \frac{\theta}{12}t_1{}^4\right) + b_1\left(\frac{t_1{}^3}{3} + \frac{\theta}{15}t_1{}^5\right). \tag{13}$$

(neglecting the higher power of θ as $0 < \theta < 1$)
The total shortage quantity during the interval $[\,t_1, T]$, say B_T , is

$$B_T = -\int_{t_1}^{T} I(t)\, dt$$

$$= -\int_{t_1}^{\mu_1}\left[a_1(t_1 - t) + \frac{b_1}{2}(t_1{}^2 - t^2)\right] dt - \int_{\mu_1}^{\mu_2}\left[a_1 t_1 - D_0 t + \frac{b_1}{2}(t_1{}^2 + \mu_1{}^2)\right] dt$$

$$- \int_{\mu_2}^{T}\left[a_1 t_1 - a_2 t + \frac{b_2}{2}(t^2 + \mu_2{}^2) + \frac{b_1}{2}(t_1{}^2 + \mu_1{}^2)\right] dt$$

$$= \frac{a_1}{2}(t_1 - \mu_1)(t_1 + \mu_1 - 2T) + \frac{b_1}{6}(2t_1{}^3 - 2\mu_1{}^3 + 3T\mu_1{}^2 - 3Tt_1{}^2) +$$
$$\frac{a_2}{2}(\mu_2 - T)^2 + \frac{b_2}{6}(3T\mu_2{}^2 - T^3 - 2\mu_2{}^3) + \frac{D_0}{2}(\mu_1 - \mu_2)(\mu_1 + \mu_2 - 2T)\ . \tag{14}$$

Then, the average total cost per unit time under the condition $t_1 \le \mu_1$ can be given by

$$C_1(t_1) = \frac{1}{T}[A_0 + c_1 D_T + c_2 H_T + c_3 B_T]. \tag{15}$$

The first order differential of $C_1(t_1)$ with respect to t_1 is as follows:

$$\frac{dC_1(t_1)}{dt_1} =$$

ISBN - 978-93-81583-56-2

$\frac{1}{T}\left[c_1\frac{\theta t_1{}^2}{2}+c_2\left(t_1+\frac{\theta}{3}t_1{}^3\right)+c_3(t_1-T)\right](a_1+b_1t_1)$ (16)

The necessary condition for $C_1(t_1)$ in (15) to be minimized is $\frac{dC_1(t_1)}{dt_1}=0$, that is

$$\left[c_1\frac{\theta t_1{}^2}{2}+c_2\left(t_1+\frac{\theta}{3}t_1{}^3\right)+c_3(t_1-T)\right](a_1+b_1t_1)=0. \quad . \tag{17}$$

Let $(t_1)=c_1\frac{\theta t_1{}^2}{2}+c_2\left(t_1+\frac{\theta}{3}t_1{}^3\right)+c_3(t_1-T)$,

Since $(0)=-c_3T<0$, $f(T)=c_1\frac{\theta T^2}{2}+c_2\left(T+\frac{\theta}{3}T^3\right)>0$ and $f'(t_1)=c_1\theta t_1+c_2(1+3\theta t_1{}^2)+c_3>0$, it implies that $f(t_1)$ is a strictly monotone increasing function and Eq. (17) has a unique solution as t_1^* , for $t_1^*\in(0,T)$. Therefore, we have

Property 1. The deteriorating inventory model under the condition $0<t_1\le\mu_1$, $C_1(t_1)$ obtains its minimum at $t_1=t_1{}^*$, where $f(t_1{}^*)=0$ if $t_1{}^*<\mu_1$. On the other hand, $C_1(t_1)$ obtains its minimum at $t_1{}^*=\mu_1$ if $t_1{}^*\ge\mu_1$.

From property 1, we know that the total back-order amount at the end of the cycle is $\Delta_1=a_1(\mu_1-t_1^*)+\frac{b_1}{2}\left(\mu_1{}^2-t_1^{*2}\right)+D_0(\mu_2-\mu_1)+a_2(T-\mu_2)-\frac{b_2}{2}(T^2-\mu_2{}^2)$. Therefore, the optimal order quantity, denoted by Q^*, is $Q^*=S^*+\Delta_1$, where S^* denotes the optimal value of S.

Case 2. $\mu_1\le t_1\le\mu_2$.

If the time $t_1\in(\mu_1,\mu_2)$, then, the differential equations governing the inventory model can be expressed as follows:

$$\frac{dI(t)}{dt}=-\theta tI(t)-(a_1+b_1t),\qquad 0<t<\mu_1, \tag{18}$$

$$\frac{dI(t)}{dt}=-\theta tI(t)-D_0,\ \mu_1<t<t_1\ , \tag{19}$$

$$\frac{dI(t)}{dt}=-D_0,\qquad t_1<t<\mu_2 \tag{20}$$

and

$$\frac{dI(t)}{dt}=-(a_2-b_2t),\quad \mu_2<t<T\ . \tag{21}$$

Solving the differential equations (18)-(21) with $I(t_1)=0$ and neglecting the higher power of θ as $0<\theta<1$, we have

$$I(t) = \begin{bmatrix} \left\{ D_0\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_1\left(\frac{{\mu_1}^2}{2} + \frac{\theta}{24}{\mu_1}^4\right)\right\} \\ -\left\{a_1\left(t + \frac{\theta}{6}t^3\right) + b_1\left(\frac{t^2}{2} + \frac{\theta}{8}t^4\right)\right\} \end{bmatrix} e^{-\frac{\theta t^2}{2}} \quad 0 \le t \le \mu_1 \tag{22}$$

$$I(t) = D_0\left[\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - \left(t + \frac{\theta}{6}t^3\right)\right] e^{-\frac{\theta t^2}{2}}, \mu_1 \le t \le t_1 \ . \tag{23}$$

$$I(t) = D_0(t_1 - t)\ , \qquad t_1 \le t \le \mu_2\ , \tag{24}$$

$$I(t) = D_0 t_1 - a_2 t + \frac{b_2}{2}(t^2 + {\mu_2}^2),\ \ \mu_2 \le t \le T\ . \tag{25}$$

The beginning inventory level can be computed as

$$S = I(0) = D_0\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_1\left(\frac{{\mu_1}^2}{2} + \frac{\theta}{24}{\mu_1}^4\right) \tag{26}$$

The total number of items which perish in the interval $[0, t_1]$ is

$$D_T = S - \int_0^{t_1} D(t)\,dt$$

$$= S - \left[\int_0^{\mu_1} D(t)\,dt + \int_{\mu_1}^{t_1} D(t)\,dt\right]$$

$$= S - \int_0^{\mu_1}(a_1 + b_1 t)\,dt - \int_{\mu_1}^{t_1} D_0\,dt = \frac{\theta}{6}\left(D_0 t_1^3 - \frac{b_1}{4}{\mu_1}^4\right). \tag{27}$$

The total number of inventory carried during the interval $[0, t_1]$ is

$$H_T = \int_0^{t_1} I(t)\,dt$$

$$= \int_0^{\mu_1}\left[\begin{bmatrix} \left\{ D_0\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_1\left(\frac{{\mu_1}^2}{2} + \frac{\theta}{24}{\mu_1}^4\right)\right\} \\ -\left\{a_1\left(t + \frac{\theta}{6}t^3\right) + b_1\left(\frac{t^2}{2} + \frac{\theta}{8}t^4\right)\right\} \end{bmatrix} e^{-\frac{\theta t^2}{2}}\right] dt$$

$$+ \int_{\mu_1}^{t_1}\left[D_0\left[\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - \left(t + \frac{\theta}{6}t^3\right)\right] e^{-\frac{\theta t^2}{2}}\right] dt$$

$$= D_0\left(\frac{{t_1}^2}{2} + \frac{\theta}{12}{t_1}^4\right) - b_1\left(\frac{{\mu_1}^3}{6} + \frac{\theta}{60}{\mu_1}^5\right) \tag{28}$$

(neglecting the higher power of θ as $0 < \theta < 1$)

The total shortage quantity during the interval $[\ t_1, T]$ is

$$B_T = -\int_{t_1}^{T} I(t)\,dt$$

ISBN - 978-93-81583-56-2

$$= -\int_{t_1}^{\mu_2}[D_0(t_1 - t)]\,dt - \int_{\mu_2}^{T}\left[D_0 t_1 - a_2 t + \frac{b_2}{2}(t^2 + \mu_2{}^2)\right]dt$$

$$= \frac{D_0}{2}(\mu_2 - t_1)^2 + \frac{a_2}{2}(T - \mu_2)^2 + \frac{b_2}{6}(3T\mu_2{}^2 - T^3 - 2\mu_2{}^3)$$

$$+D_0(\mu_2 - t_1)(T - \mu_2)\,. \tag{29}$$

Then, the average total cost per unit time under the condition $\mu_1 \le t_1 \le \mu_2$ can be given by

$$C_2(t_1) = \frac{1}{T}[A_0 + c_1 D_T + c_2 H_T + c_3 B_T] \tag{30}$$

The first order differential of $C_2(t_1)$ with respect to t_1 is as follows:

$$\frac{dC_2(t_1)}{dt_1} = \frac{D_0}{T}\left[c_1\frac{\theta t_1{}^2}{2} + c_2\left(t_1 + \frac{\theta}{3}t_1{}^3\right) + c_3(t_1 - T)\right] \tag{31}$$

The necessary condition for $C_2(t_1)$ in (15) to be minimized is $\frac{dC_2(t_1)}{dt_1} = 0$, that is

$$\left[c_1\frac{\theta t_1{}^2}{2} + c_2\left(t_1 + \frac{\theta}{3}t_1{}^3\right) + c_3(t_1 - T)\right] = 0 \tag{32}$$

Similar to the first case, we have

Property 2. The inventory model under the condition $\mu_1 < t_1 \le \mu_2$, $C_2(t_1)$ obtains its minimum at $t_1 = t_1{}^*$, where $f(t_1{}^*) = 0$ if $\mu_1 < t_1{}^* < \mu_2$. $C_2(t_1)$ obtains its minimum at $t_1{}^* = \mu_1$ if $t_1{}^* < \mu_1$ and $C_2(t_1)$ obtains its minimum at $t_1{}^* = \mu_2$ if $\mu_2 < t_1{}^*$.

From property 2, we know that the total back-order amount at the end of the cycle is $\Delta_2 = D_0(\mu_2 - t_1{}^*) + a_2(T - \mu_2) - \frac{b_2}{2}(T^2 - \mu_2{}^2)$. Therefore, the optimal order quantity, denoted by Q^*, is $Q^* = S^* + \Delta_2$, where S^* denotes the optimal value of S.

Case 3. $\mu_2 \le t_1 < T$.

If the time $t_1 \in [\mu_2, T)$, then, the differential equations governing the inventory model can be expressed as follows:

$$\frac{dI(t)}{dt} = -\theta t I(t) - (a_1 + b_1 t), \qquad 0 < t < \mu_1\,, \tag{33}$$

$$\frac{dI(t)}{dt} = -\theta t I(t) - D_0, \qquad \mu_1 < t < \mu_2\,, \tag{34}$$

$$\frac{dI(t)}{dt} = -\theta t I(t) - (a_2 - b_2 t), \quad \mu_2 < t < t_1 \tag{35}$$

and

$$\frac{dI(t)}{dt} = -(a_2 - b_2 t), \quad t_1 < t < T\,. \tag{36}$$

Solving the differential equations (33)-(36) with $I(t_1) = 0$ and neglecting the higher power of θ as $0 < \theta < 1$, we have

$$I(t) = \left[\begin{array}{c} \left\{\begin{array}{c} a_2\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_2\left(\frac{{t_1}^2}{2} + \frac{\theta}{8}{t_1}^4\right) \\ -b_1\left(\frac{{\mu_1}^2}{2} + \frac{\theta}{24}{\mu_1}^4\right) - b_2\left(\frac{{\mu_2}^2}{2} + \frac{\theta}{24}{\mu_2}^4\right) \end{array}\right\} \\ -\left\{a_1\left(t + \frac{\theta}{6}t^3\right) + b_1\left(\frac{t^2}{2} + \frac{\theta}{8}t^4\right)\right\} \end{array}\right] e^{-\frac{\theta t^2}{2}}. \qquad 0 \le t \le \mu_1\,, \tag{37}$$

$$I(t) = \left[\begin{array}{c} a_2\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_2\left(\frac{{t_1}^2}{2} + \frac{\theta}{8}{t_1}^4\right) \\ -b_2\left(\frac{{\mu_2}^2}{2} + \frac{\theta}{24}{\mu_2}^4\right) - D_0\left(t + \frac{\theta}{6}t^3\right) \end{array}\right] e^{-\frac{\theta t^2}{2}} \quad , \qquad .\mu_1 \le t \le \mu_2\,, \tag{38}$$

$$I(t) = \left[\begin{array}{c} \left\{a_2\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_2\left(\frac{{t_1}^2}{2} + \frac{\theta}{8}{t_1}^4\right)\right\} \\ -\left\{a_2\left(t + \frac{\theta}{6}t^3\right) - b_2\left(\frac{t^2}{2} + \frac{\theta}{8}t^4\right)\right\} \end{array}\right] e^{-\frac{\theta t^2}{2}}, \qquad .\,\mu_2 \le t \le t_1\,, \quad 39)$$

$$I(t) = a_2(t_1 - t) + \frac{b_2}{2}(t^2 - {t_1}^2),\ t_1 \le t \le T\,. \tag{40}$$

The beginning inventory level can be computed as

$$S = I(0) = a_2\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_2\left(\frac{{t_1}^2}{2} + \frac{\theta}{8}{t_1}^4\right) - b_1\left(\frac{{\mu_1}^2}{2} + \frac{\theta}{24}{\mu_1}^4\right) - b_2\left(\frac{{\mu_2}^2}{2} + \right.$$
$$\theta 24 \mu 24\,. \tag{41}$$

The total number of items which perish in the interval $[0, t_1]$ is

$$D_T = S - \int_0^{t_1} D(t)\,dt$$
$$= S - \left[\int_0^{\mu_1} D(t)\,dt + \int_{\mu_1}^{\mu_2} D(t)\,dt + \int_{\mu_2}^{t_1} D(t)\,dt\right]$$
$$= S - \int_0^{\mu_1}(a_1 + b_1 t)\,dt - \int_{\mu_1}^{\mu_2} D_0\,dt - \int_{\mu_2}^{t_1}(a_2 - b_2 t)\,dt$$
$$= a_2\frac{\theta t_1^3}{6} - b_2\left(\frac{\theta}{8}t_1^4 + \frac{\theta}{24}{\mu_2}^4\right) - b_1\frac{\theta}{24}{\mu_2}^4 \tag{42}$$

The total number of inventory carried during the interval $[0, t_1]$ is

$$H_T = \int_0^{t_1} I(t)\,dt$$

$$= \int_0^{\mu_1} \left[\begin{Bmatrix} a_2\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_2\left(\frac{{t_1}^2}{2} + \frac{\theta}{8}{t_1}^4\right) \\ -b_1\left(\frac{{\mu_1}^2}{2} + \frac{\theta}{24}{\mu_1}^4\right) - b_2\left(\frac{{\mu_2}^2}{2} + \frac{\theta}{24}{\mu_2}^4\right) \end{Bmatrix} \\ -\left\{a_1\left(t + \frac{\theta}{6}t^3\right) + b_1\left(\frac{t^2}{2} + \frac{\theta}{8}t^4\right)\right\} \right] e^{-\frac{\theta t^2}{2}}\, dt$$

$$+ \int_{\mu_1}^{\mu_2} \begin{bmatrix} a_2\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_2\left(\frac{{t_1}^2}{2} + \frac{\theta}{8}{t_1}^4\right) \\ -b_2\left(\frac{{\mu_2}^2}{2} + \frac{\theta}{24}{\mu_2}^4\right) - D_0\left(t + \frac{\theta}{6}t^3\right) \end{bmatrix} e^{-\frac{\theta t^2}{2}}\, dt$$

$$+ \int_{\mu_2}^{t_1} \begin{bmatrix} \left\{a_2\left(t_1 + \frac{\theta}{6}{t_1}^3\right) - b_2\left(\frac{{t_1}^2}{2} + \frac{\theta}{8}{t_1}^4\right)\right\} \\ -\left\{a_2\left(t + \frac{\theta}{6}t^3\right) - b_2\left(\frac{t^2}{2} + \frac{\theta}{8}t^4\right)\right\} \end{bmatrix} e^{-\frac{\theta t^2}{2}}\, dt$$

$$= a_2\left(\frac{{t_1}^2}{2} + \frac{\theta}{12}{t_1}^4\right) - b_2\left(\frac{{t_1}^3}{3} + \frac{\theta}{15}{t_1}^5\right) - b_1\left(\frac{{\mu_1}^3}{6} + \frac{\theta}{60}{\mu_1}^5\right) - b_2\left(\frac{{\mu_2}^3}{6} + \frac{\theta}{60}{\mu_2}^5\right). \quad (43)$$

(neglecting the higher power of θ as $0 < \theta < 1$)

The total shortage quantity during the interval $[\, t_1, T]$ is

$$B_T = -\int_{t_1}^{T} I(t)\, dt$$

$$= -\int_{t_1}^{T} \left[a_2(t_1 - t) + \frac{b_2}{2}(t^2 - {t_1}^2)\right] dt$$

$$= \frac{a_2}{2}(T - t_1)^2 + \frac{b_2}{2}t_1^2(T - t_1) + \frac{b_2}{6}({t_1}^3 - T^3)\,. \quad (44)$$

Then, the average total cost per unit time under the condition $\mu_2 \le t_1 \le T$ can be given by

$$C_3(t_1) = \frac{1}{T}[A_0 + c_1 D_T + c_2 H_T + c_3 B_T]\,. \quad (45)$$

The first order differential of $C_3(t_1)$ with respect to t_1 is as follows:

$$\frac{dC_3(t_1)}{dt_1} = \frac{a_2 - b_2 t_1}{T}\left[c_1\frac{\theta {t_1}^2}{2} + c_2\left(t_1 + \frac{\theta}{3}{t_1}^3\right) + c_3(t_1 - T)\right]. . \quad (46)$$

The necessary condition for $C_3(t_1)$ in (45) to be minimized is $\frac{dC_3(t_1)}{dt_1} = 0$, that is

$$\left[c_1\frac{\theta {t_1}^2}{2} + c_2\left(t_1 + \frac{\theta}{3}{t_1}^3\right) + c_3(t_1 - T)\right] = 0\,. \quad (47)$$

Similar to the first case, we have

Property 3. The inventory model under the condition $\mu_2 \le t_1 < T$, $C_3(t_1)$ obtains its minimum at $t_1 = t_1{}^*$, where $f(t_1{}^*) = 0$ if $\mu_2 < t_1{}^*$. On the other hand, $C_3(t_1)$ obtains its minimum at $t_1{}^* = \mu_2$ if $t_1{}^* < \mu_2$.

From property 3, we know that the total back-order amount at the end of the cycle is $\Delta_3 = a_2(T - t_1{}^*) - \frac{b_2}{2}\left(T^2 - t_1{}^{*2}\right)$. Therefore, the optimal order quantity, denoted by Q^*, is $Q^* = S^* + \Delta_3$, where S^* denotes the optimal value of S.

Remark 1. The previous analysis shows that Eqs. (16), (31) and (46) can be express as $\frac{dC_j(t)}{dt_1} = \frac{D_j(t_1)}{T} f(t_1), j = 1, 2, 3,$ which denotes the total marginal cost function under different demand rate conditions, respectively.

Combining the above properties, we know that $C_1(\mu_1) = C_2(\mu_1)$ and $C_2(\mu_2) = C_3(\mu_2)$. Therefore, we can derive the following result.

Theorem 1. For the deteriorating inventory model with trapezoidal type demand rate, the optimal replenishment time is $t_1{}^*$ and $C_1(t_1)$ obtains its minimum at $t_1 = t_1{}^*$, if and only if $t_1{}^* < \mu_1$. On the other hand, $C_2(t_1)$ obtains its minimum at $t_1{}^*$ if and only if $\mu_1 < t_1{}^* < \mu_2$ and $C_3(t_1)$ obtains its minimum at $t_1{}^*$ if and only if $\mu_2 < t_1{}^*$, where $t_1{}^*$ is the unique solution of $f(t_1) = 0$.

4. NUMERICAL EXAMPLES

In this section, we provide several numerical examples to illustrate the above theory.

Example 1. The parameter values are given as follows: $T = 12$ weeks, $\mu_1 = 3$ weeks, $\mu_2 = 8$ weeks, $a_1 = 100$ unit, $b_1 = 5$ unit, $a_2 = 220$ unit, $b_2 = 10$ unit, $\theta = 0.2$, $A_0 = \$200$, $c_1 = \$3$ per unit, $c_2 = \$10$ unit, $c_3 = \$5$ per unit. Based on the solution procedure as above, we have $f(\mu_1) = 5.7 > 0$, then it yields that the optimal replenishment time $t_1{}^* = 2.831137$ weeks, the optimal order quantity, Q^*, for each ordering cycle, is 2136.174 unit and the minimum cost $C_1(t_1{}^*) = \$4383.28$.

Example 2. The parameter values are given as follows: $T = 12$ weeks, $\mu_1 = 3$ weeks, $\mu_2 = 8$ weeks, $a_1 = 100$ unit, $b_1 = 5$ unit, $a_2 = 220$ unit, $b_2 = 10$ unit, $\theta = 0.2$, $A_0 = \$200$, $c_1 = \$3$ per unit, $c_2 = \$10$ unit, $c_3 = \$5$ per unit. Based on the solution procedure as above, we have $f(\mu_1) = -23.4667 < 0$ and $f(\mu_2) = 420.533 > 0$, then it yields that the optimal replenishment time $t_1{}^* = 2.831137$ weeks, the optimal order quantity, Q^*, is 2618.442 unit and the minimum cost $C_2(t_1{}^*) = \$5038.77$.

Example 3. The parameter values are given as follows: $T = 12$ weeks, $\mu_1 = 2$ weeks, $\mu_2 = 5$ weeks, $a_1 = 100$ unit, $b_1 = 4$ unit, $a_2 = 112$ unit, $b_2 = 1$ unit, $\theta = 0.4$, $A_0 = \$200$, $c_1 = \$3$ per unit, $c_2 = \$2$ unit, $c_3 = \$10$ per unit. Based on the solution procedure as above, we have $f(\mu_2) = -11.6667 < 0$, then it yields that the optimal replenishment time ${t_1}^* = 5.2962137$ weeks, the optimal order quantity, Q^*, is 2309.909 unit and the minimum cost $C_3({t_1}^*) = \$2954.1$.

5. SENSITIVITY ANALYSIS

We now study the effects of changes in the values of the system parameters T, μ_1, μ_2, a_1, b_1, a_2, b_2, θ, A_0, c_1, c_2 and c_3 on the optimal total cost and number of reorder. The sensitivity analysis is performed by changing each of the parameters by 50 %, 10 %, -10 % and -50 % taking one parameter at a time and keeping the remaining parameters unchanged.

The analysis is based on the Example 3 and the results are shown in Table 1. The following points are observed.

I. ${t_1}^*$, Q^* and $C_3({t_1}^*)$ increase with the increase in the value of the parameter T. ${t_1}^*$, Q^* and $C_3({t_1}^*)$ are all high sensitive to changes in T.

II. ${t_1}^*$ remains constant while Q^* and $C_3({t_1}^*)$ decrease with the increase in the value of the parameter μ_1. t_1^* is insensitive to changes in μ_1 while both Q^* and $C_3({t_1}^*)$ are very low sensitive to changes in μ_1 .

III. ${t_1}^*$ remains constant while Q^* and $C_3({t_1}^*)$ decrease with the increase in the value of the parameter μ_2. t_1^* is insensitive to changes in μ_2 while both Q^* and $C_3({t_1}^*)$ are very low sensitive to changes in μ_2 .

IV. ${t_1}^*$, Q^* and $C_3({t_1}^*)$ remain constant with the increase in the value of the parameter a_1. t_1^* , Q^* and $C_3({t_1}^*)$ are all insensitive to changes in a_1.

V. ${t_1}^*$, Q^* and $C_3({t_1}^*)$ remain constant with the increase in the value of the parameter b_1. t_1^* , Q^* and $C_3({t_1}^*)$ are all insensitive to changes in b_1.

VI. ${t_1}^*$ remains constant while Q^* and $C_3({t_1}^*)$ increase with the increase in the value of the parameter a_2. t_1^* is insensitive to changes in a_2 while both Q^* and $C_3({t_1}^*)$ are high sensitive to changes in a_2.

VII. ${t_1}^*$ remains constant while Q^* and $C_3({t_1}^*)$ decrease with the increase in the value of the parameter b_2. t_1^* is insensitive to changes in b_2 while both Q^* and $C_3({t_1}^*)$ are moderately sensitive to changes in b_2.

VIII. ${t_1}^*$ decreases while Q^* and $C_3({t_1}^*)$ increase with the increase in the value of the parameter θ. t_1^*, Q^* and $C_3({t_1}^*)$ are moderately sensitive to changes in θ.

IX. ${t_1}^*$ and Q^* remain constant while $C_3({t_1}^*)$ decreases with the increase in the value of the parameter A_0. t_1^* and Q^* are insensitive to changes in A_0 while $C_3({t_1}^*)$ is very low sensitive to changes in A_0.

X. ${t_1}^*$ and Q^* decrease while $C_3({t_1}^*)$ increase with the increase in the value of the parameter c_1. t_1^* is very low sensitive to changes in c_1 while both Q^* and $C_3({t_1}^*)$ are moderately sensitive to changes in c_1.

XI. $t_1{}^*$ and Q^* decrease while $C_3(t_1{}^*)$ increase with the increase in the value of the parameter c_2. t_1^*, Q^* and $C_3(t_1{}^*)$ are moderately sensitive to changes in c_2.

XII. $t_1{}^*$, Q^* and $C_3(t_1{}^*)$ increase with the increase in the value of the parameter c_3. t_1^*, Q^* and $C_3(t_1{}^*)$ are moderately sensitive to changes in c_3.

Table 1.

Parameter	**% Change in Parameter**	t_1^*	Q^*	**% Change in Q^***	$C_3(t_1^*)$	**% Change in $C_3(t_1^*)$**
T	+50	6.563309	3840.27	+66.252	5023.77	+70.0609
	+10	5.580624	2608.502	+12.9266	3355.8	+13.5981
	-10	4.99122	2016.689	-12.694	2561.56	-13.288
	-50	3.469753	926.674	-59.8827	1134.48	-61.5964
μ_1	+50	5.2962137	2295.579	-0.620371	2949.97	-0.139806
	+10	5.2962137	2307.739	-0.0939431	2953.59	-0.0172641
	-10	5.2962137	2311.799	+0.0818214	2954.49	+0.013202
	-50	5.2962137	2316.909	+0.303042	2955.26	+0.0392675
μ_2	+50	5.2962137	2251.969	-2.50832	2912.38	-1.41227
	+10	5.2962137	2302.449	-0.322956	2949.62	-0.151654
	-10	5.2962137	2315.869	+0.258019	2957.36	+0.110355
	-50	5.2962137	2329.049	+0.828604	2962.94	+0.299245
a_1	+50	5.2962137	2309.909	0	2954.1	0
	+10	5.2962137	2309.909	0	2954.1	0
	-10	5.2962137	2309.909	0	2954.1	0
	-50	5.2962137	2309.909	0	2954.1	0
b_1	+50	5.2962137	2305.379	-0.196112	2953.45	-0.0220033
	+10	5.2962137	2309.009	-0.0389626	2953.97	-0.00440066
	-10	5.2962137	2310.819	+0.0393955	2954.23	+0.00440066
	-50	5.2962137	2314.449	+0.196545	2954.75	+0.0220033
a_2	+50	5.2962137	3536.53	+53.1026	4517.05	+52.9078
	+10	5.2962137	2555.231	+10.6204	3266.69	+10.5755
	-10	5.2962137	2064.587	-10.6204	2641.51	-10.5816
	-50	5.2962137	1083.294	-53.1023	1391.15	-52.9078
b_2	+50	5.2962137	2242.781	-2.90609	2860.51	-3.16814
	+10	5.2962137	2296.481	-0.581322	2935.38	-0.633696
	-10	5.2962137	2323.337	+0.581322	2972.82	+0.633696
	-50	5.2962137	2377.037	+2.90609	3047.69	+3.16814
θ	+50	4.746324	2393.756	+3.62988	3252.71	+10.1083
	+10	5.164888	2331.351	+0.928521	3024.02	+2.36688
	-10	5.442604	2285.079	-1.07493	2877.2	-2.60316
	-50	6.273193	2130.689	-7.75875	2462.27	-16.6491
A_0	+50	5.2962137	2309.909	0	2962.43	+0.281981
	+10	5.2962137	2309.909	0	2955.77	+0.0565316
	-10	5.2962137	2309.909	0	2952.43	-0.0565316
	-50	5.2962137	2309.909	0	2945.77	-0.281981
C_1	+50	5.100415	2196.881	-4.89318	3079.13	+4.23242
	+10	5.255396	2285.665	-1.04957	2980.25	+0.88521
	-10	5.33914	2335.06	+1.08883	2927.33	-0.906198
	-50	5.514112	2445.642	+5.87612	2813.58	-4.75678
	+50	4.790058	2034.326	-11.9305	3257.1	+10.2569

ISBN - 978-93-81583-56-2

C_2	+10 -10 -50	5.178421 5.425202 6.109325	2240.915 2388.974 2872.938	-2.98687 +3.42286 +24.3745	3023.14 2879.55 2502.98	+2.33709 -2.52361 -15.271
C_3	+50 +10 -10 -50	5.977307 5.453714 5.124195 4.209897	2770.766 2406.956 2210.169 1781.421	+19.9513 +4.20133 -4.31792 -22.8792	3828.54 3145.0 2753.38 1821.96	+29.6009 +6.46221 -6.79462 -38.3244

6. CONCLUSION

The inventory models for the deteriorating items having increasing-steady-decreasing demand pattern have been presented in this paper. The deterioration rate is presented by a variable deterioration rate. The demand is represented by trapezoidal type demand rate that describes the increasing, the steady and the declining phases of demand commonly experienced by many seasonal products in the market. The demand rate is a piecewise linearly functions. An optimal replenishment policy of inventory model is proposed. The model considered above is suitable for items having variable demand rate, earlier models have considered deteriorating items having constant rate of deterioration. This model can be used for the seasonal items like fruits, vegetables etc., whose deterioration rate increase with time. Demand pattern considered here is trapezoidal type demand. For the market information, we can find that trapezoidal type demand rate along with variable rate of deterioration is more applicable than the trapezoidal type demand rate or ramp type demand rate model in the stage of product life cycle. Cheng and Wang [22] considered the trapezoidal type demand rate and in this paper, we extended it to the trapezoidal type demand rate with variable deterioration rate. This model can be further extended for items having trapezoidal type demand rate with two-parameter Weibull distribution deterioration or three-parameter Weibull distribution deterioration. It can also be extended foe items having quadratic-constant-quadratic demand rate with Weibull distribution deterioration.

7. ACKNOWLEDGMENT

The authors greatly appreciate the helpful comments and suggestions of the reviewers.

8. REFERENCES

1. P.M. Ghare, G.F. Schrader, “A model for exponentially decaying inventories”, J. Ind. Eng., vol. 14, 1963, pp. 238–243.
2. R. B. Covert, G. S. Philip, “An EOQ model with Weibull distribution deterioration”, AIIE Transaction, vol. 5, 1973, pp. 323-326.
3. R. B. Misra, “Optimal production lot-size model for a system with deteriorating inventory”, Journal of operational research society, vol. 36, 1975, pp. 495-505.
4. E. A. Silver, H. C. Meal, “A simple modification of the EOQ for the case of a varying demand arte”, Production and Inventory Management, vol. 10 (4), 1969, pp. 52-65.
5. W.A. Donaldson, “Inventory replenishment policy for a linear trend in demand: an analytic solution”, Oper. Res. Quarter, vol. 28, 1977, pp. 663–670.

6. E. A. Silver, "A simple inventory replenishment decision rule for a linear trend in demand", Journal of operational research society, vol. 30, 1979, pp. 71-75.
7. E. Ritchie, "Practical inventory replenishment policies for a linear trend in demand followed by a period of study demand", Journal of operational research society, vol. 31, 1980, pp. 605-613.
8. E. Ritchie, "The EOQ for linear increasing demand, a simple optimal solution", Journal of operational research society, vol. 35, 1984, pp. 949-952.
9. R. M. Hill, "Inventory model for increasing demand followed by level demand", Journal of operational research society, vol. 46, 1995, pp. 1250-1259.
10. B. Mandal, A.K. Pal, "Order level inventory system with ramp type demand rate for deteriorating items", J. Interdisciplinary Math., vol. 1, 1998, pp. 49–66.
11. K.S. Wu, L.Y. Ouyang, "A replenishment policy for deteriorating items with ramp type demand rate (Short Communication)", Proc. Nat. Sci. Council ROC (A), vol. 24, 2000, pp. 279–286.
12. Deng Peter Shaohua, H.-J. Lin Robert, Chu Peter, "A note on the inventory models for deteriorating items with ramp type demand rate", European Journal of Operational Research, vol. 178, 2007, pp. 112-120.
13. K. S. Wu, "An EOQ model for item s with weibull distribution deterioration, ramp type demand rate and partial backlogging", Production and planning control, vol. 12, 2001, pp. 787-793.
14. K. Skouri, I. Konstantaras, S. Papachristos, I. Ganas, "Inventory models with ramp type demand rate, partial backlogging and weibull deterioration rate", European J. Oper. Res., vol. 192, 2009, pp. 79–92.
15. B. C. Giri, A. K. Jalan & K. S. Chaudhuri, "Economic order quantity model with Weibull deterioration distribution, shortage and ramp-type demand", International Journal of Systems Science, vol. 34, 2003, pp. 237–243.
16. J.W. Wu, C. Lin, B. Tan, W.C. Lee, "An EOQ model with ramp type demand rate for items with Weibull deterioration", Int. J. Inform. Management Sci., vol. 10, 1999, pp. 41–51.
17. S.K. Manna, K.S. Chaudhuri, "An EOQ model with ramp type demand rate, time dependent deteriorating rate, unit production cost and shortages", European J. Oper. Res., vol. 171, 2006, pp. 557–566.
18. L. Benkherouf, "On an inventory model with deteriorating items and decreasing time-varying demand and shortages", European J. Oper. Res., vol. 86, 1995, pp. 293–299.
19. A. Goswami, K. S. Chaudhuri, "An EOQ model for deteriorating items with shortages and a linear trend in demand", Journal of the Operational Research Society, vol. 42, 1991, pp. 1105–1110.
20. S.Panda, S. Senapati, & M. Basu, "Optimal replenishment policy for perishable seasonal products in a season with ramp-type time dependent demand", Computers & Industrial Engineering, vol. 54, 2008, pp. 301–314.
21. B.C. Giri, A.K. Jalan, K.S. Chaudhuri, "Economic order quantity model with weibull deterioration distribution, shortage and ramp-type demand", Int. J.Syst. Sci., vol. 34, 2003, pp. 237–243.
22. M. Cheng, G. Wang, "A note on the inventory model for deteriorating items with trapezoidal type demand rate", Comput. Ind. Eng., vol. 56, 2009, pp. 1296–1300.

* * * *

[1]*Department of Mathematics,*
C. V. Raman College of engineering, Bhubaneswar, Odisha, India
e.mail: trailokyanaths108@gmail.com
[2]*Department of Statistics,*
Sambalpur University, Odisha, India