DIRECT LIMIT OF DIRECT SYSTEM OF P-NEAR-RINGS

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***Abstract:** In this paper first we present a sheaf of rings over a topological pace X, ringed spaces, partition property of a Boolean space (totally disconnected compact hausdorff topological space), theorems related to sheaves of rings over a Boolean space, in a ringed space over Boolean space, the ring of global sections is commutative if each stalk is commutative, one-one correspondence between the class of completely prime ideals of a near-ring N and prime ideals of its Boolean ring B(N). We prove that a p-near ring N with identity Spec N and Spec B(N) are homeomorphic, we introduce the sheaf of p-near-rings with identity, at the end we prove every p-near-ring with identity is isomorphic with a sheaf Σ of p-near- rings with identity over a Boolean space X such that (X, N) is reduced.*

Keywords : Glueing condition, Spec N, Spec B(N), sheaf Σ of p-near- rings

1.1 **Definition:** Let X be a topological space. A sheaf F of sets on X if it satisfies :

I. (a) For each open set U of X, a set F(U)

(called the set of sections of F over U).

(b) For each pair of open sets $V \subseteq U$ of X, a restriction map $\rho_v^u : F(U) \to F(V)$ such that

(i) for all U, $\rho_u^u = id_u$.

(ii) whenever $W \subseteq V \subseteq U$ (all open) $\rho_w^u = \rho_w^v \circ \rho_v^u$.

II. If $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ is open covering of U and s, $s^l \in F(U)\ \forall \lambda \in \Lambda$,

$\rho_{u_\lambda}^u (s) = \rho_{u_\lambda}^u (s^l)$ then $s = s^l$.

III. If U is open in x, $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ is an open covering of U and $(S_\lambda)_{\lambda\in\Lambda}$,

is a family of sections of F with $\forall \lambda \in \Lambda$, $s_\lambda \in F(U_\lambda)$, such that $\forall \lambda, u$, $\rho^{u_\lambda}_{u_\lambda \cap u_u} (s_\lambda) = \rho^{u_u}_{u_\lambda \cap u_u} (s_u)$, then there is $s \in F(U)$ such that λ

$\rho_{u_\lambda}^u (s) = s_\lambda$.

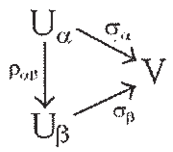
1.2 **Note:** If F satisfies only I, then F is called presheaf of sets over X.

1.3 **Definition:** Suppose F is a presheaf of sets over a topological space and

$x \in X$. B_x = { U / U is open set in X and $x \in U$}. Then $F_x = \underset{U \in Bx}{\underline{\text{limit}}} F(U)$.

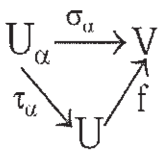
is called stalk of F at x ; this comes equipped with maps $F(U) \to Fx$: $s \to s_x$ whenever an open set $U \ni x$, the members of F_x are called germs.

1.4 **Definition:** Given a direct system in the notation of 1.1, a target for the system is a set V and a collection of maps $(\sigma_\alpha : U_\alpha \to V)_{\alpha \in \Lambda}$ satisfying the compatibility condition: $\forall \alpha \leq \beta$



commutes; i.e $\sigma_\alpha = \sigma_\beta \circ \rho_{\alpha\beta}$.

A direct limit for the system is a target U, $(\tau_\alpha : U_\alpha \to U)_{\alpha \in \Lambda}$ satisfy the universal property : for a target V (with maps σ_α as above) there is a unique map $f : U \to V$ such that $\forall \alpha \in \Lambda$



commutes.

1.5 **Proposition :** Any two direct limits for a direct system are naturally isomorphic (i.e. there is a bijection between them compatible with all the τ_α .

1.6 **Notation:** Thus there is some justification in speaking of the direct limit and denoting it by $\lim_{\alpha \in \Lambda} U_\alpha$.

1.7 **Theorem:** Suppose U, $(\tau_\alpha : U_\alpha \to U)_{\alpha \in \Lambda}$ is a target for the system $(U_\alpha)_{\alpha \in \Lambda}$, $(\rho_{\alpha\beta})_{(\alpha, \beta) \in \Lambda 1}$, such that

(i) $\forall u \in U \; \exists \; \alpha \in \Lambda$ such that $u \in Im(\tau_\alpha)$

(ii) if $\alpha, \beta \in \Lambda$ and $u_\alpha \in U_\alpha$ and $u_\beta \in U_\beta$ then $\tau_\alpha(u_\alpha) = \tau_\beta(u_\beta) \Leftrightarrow \exists \; \gamma \in \Lambda$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$ and $\rho_{\alpha\gamma}(u_\alpha) = \rho_{\beta\gamma}(u_\beta)$.

Then U is a direct limit of the system.

1.8 **Theorem:** U with the $(\tau_\alpha)_{\alpha \in \Lambda}$ is a direct limit for the system $(U_\alpha)_{\alpha \in \Lambda}$. Hence every direct system of sets has a direct limit.

Main Theorems

2.1 **Theorem:** Suppose $\{N_\alpha / \alpha \in \Delta\}(\rho_{\alpha\beta})_{(\alpha,\beta)\in\Lambda}$, is a direct system of p-near-rings with identity. Then $\underset{\alpha\in\Lambda}{\underline{\text{limit}}}\ N_\alpha$ is a p-near-ring with identity.

Proof : Suppose $M = \underset{\alpha\in\Lambda}{\cup} N_\alpha$. Define N on M by defining m ~ n for

$m \in N_\alpha$, $n \in N_\beta$ if $\exists\ r \geq \alpha, \beta$ such that $\rho_{\alpha\gamma}(m) = \rho_{\beta\delta}(n)$. Clearly ~ is reflexive and symmetric.

Suppose $m \sim n$, $n \sim s \Rightarrow m \in N_\alpha, n \in N_\beta, o \in N_\gamma$.

Since $m \sim n\ \exists\ \delta \geq \alpha, \beta \ni \rho_{\alpha\gamma}(m) = \rho_{\beta\delta}(n)$.

Since $n \sim s\ \exists\ \varepsilon \geq \beta, \gamma \ni \rho_{\beta\varepsilon}(n) = \rho_{\gamma\varepsilon}(s)$.

Let $\xi \in \Lambda$ and $\xi \geq \delta, \varepsilon$.

$\rho_{\alpha\xi}(m) = \rho_{\delta\xi}\,\rho_{\alpha\delta}(m) \quad = \rho_{\delta\xi}\,\rho_{\beta\delta}(n) \quad = \rho_{\beta\xi}(n) \quad = \rho_{\varepsilon\xi}\,\rho_{\beta\varepsilon}(n)$

$= \rho_{\varepsilon\xi}\,\rho_{\gamma\varepsilon}(s) \quad = \rho_{\gamma\xi}(s)\ \therefore\ m \sim s$

$\therefore$ ~ is an equivalence relation on M.

Let $N = M/\sim$ and $T_\alpha: N_\alpha \to N$ be the composite maps $N_\alpha \to M \to M/\sim\ = N$

$\therefore$ N with $(T_\alpha)_{\alpha\in\Lambda}$, is the direct limit of the direct system of p-near-rings $(N_\alpha)_{\alpha\in\Lambda}$, $(\rho_{\alpha\beta})_{\alpha\leq\beta}$ i.e $\mathbf{N} = \underset{\alpha\in\Lambda}{\underline{\text{limit}}}\ N_\alpha$

Claim: $(N, +, \cdot)$ is a p-near-ring. Clearly $\overline{O_\alpha} = \overline{O_\beta}$.

Since $\exists\ \gamma \geq \alpha, \beta \ni \rho_{\alpha\gamma}(O_\alpha) = \rho_{\beta\gamma}(O_\beta) = O_\gamma$.

Define $\overline{O} = O_\alpha \in N$

Define $\overline{1} = 1_\alpha \in N$

Let $n_i \in N_{\alpha i}$, $i = 1, 2\ \exists\ \gamma \geq \alpha_i$, $i = 1, 2$

Define $\overline{n_1} + \overline{n_2} = \overline{\rho_{\alpha 1\gamma}(n_1) + \rho_{\alpha 2\gamma}(n_2)}$ -------- (I)

This is clearly independent of the choice of $\overline{n_i}$ in n_i. We also have to

show that this is independent of the choice of γ.

Let $\delta \geq \alpha_1, \alpha_2$.

Choose $\varepsilon \geq \gamma$, δ

$$\rho_{\gamma\varepsilon}(\rho_{\alpha 1\gamma}(n_1) + \rho_{\alpha 2\gamma}(n_2)) = \rho_{\gamma\varepsilon}\,\rho_{\alpha 1\gamma}(n_1) + \rho_{\gamma\varepsilon}\,\rho_{\alpha 2\gamma}(n_2)$$

$$= \rho_{\alpha 1\varepsilon}(n_1) + \rho_{\alpha 2\varepsilon}(n_2)$$

III^{ly} $\rho_{\delta\varepsilon}(\rho_{\alpha 1\delta}(n_1) + \rho_{\alpha 2\delta}(n_2)) = \rho_{\alpha 1\varepsilon}(n_1) + \rho_{\alpha 2\varepsilon}(n_2)$

$\therefore$ (I) is independent of the choice of $\gamma \geq \alpha_1$, α_2

$\therefore$ $\bar{n}_1 + \bar{n}_2 \in N$

III^{ly} $\bar{n}_1 \cdot \bar{n}_2 = \overline{\rho_{\alpha 1\gamma}(n_1) \cdot \rho_{\alpha 2\gamma}(n_2)}$ for any $\gamma \geq \alpha_1$, α_2

Suppose $n \in N \Rightarrow n \in N_\alpha$.

$$\bar{n} = \overline{\rho_{\alpha\alpha}}(n)$$

$$(\bar{n})^P = \overline{\rho_{\alpha\alpha}(n)^P} = \overline{\rho_{\alpha\alpha}(n)} = \bar{n}$$

$$\overline{\rho(n)} = \overline{\rho_{\alpha\alpha}(n) + \dots\dots\dots + \rho_{\alpha\alpha}(n)} \quad \text{(p times)}$$

$$= \bar{O}_\alpha = O$$

$\therefore$ (N , + , ·) is a p-near-ring.

$\therefore$ $\underset{\alpha\in\Lambda}{\underline{\text{limit}}}\, N_\alpha$ is a p-near-ring.

2.2 **Definition:** A sheaf F of p-near-ring with identity over a topological space X is a sheaf of sets such that (a) Each F(U) has a given p-near-ring structure. (b). Every restriction map $\rho^u{}_v$ is a p-near-ring homomorphism with respect to these structures.

2.3 **Theorem :** Suppose X is a topological space. F is a sheaf p-near-ring's over X . $N_x = \underset{x\in U}{\underline{\text{limit}}}\, F(U)$, U is open in X.

Then $N = \bigcup_{x\in U} N_x$. Define $\Pi : N \to X$ by $\Pi(r) = x$ if $r \in N_x$.

Then (I). If $n \in N$, $\exists$ open sets U in N with $n \in U$ and $M \subseteq X$ such that Π maps U homeomorphically on to M.

(II). $n \to -n$ is continuous on N to N. And the maps $(n, m) \to n + m$,

$(n, m) \to n \cdot m$ are continuous on $N + N = \{(n, m) / \Pi(n) = \Pi(m)\}$ with subspace topology in $N \times N$.

(III). The mapping $x \to 1_x$ is continuous on X to N.

Proof: $+ : N + N \rightarrow N$. Suppose U is open set in N

$\Rightarrow$ $U \times U$ is open in $N \times N$

$\Rightarrow$ $U^{I} = (U \times U) \cap (N + N)$ is open in $N + N$

and $U^{I} \subseteq +^{-1}(U)$ i.e

$\Rightarrow$ $U^{I} + U^{I} \subseteq U$.

$\therefore$ + is continuous.

lll^{ly} $\cdot$ is continuous and $X \rightarrow N$ is continuous & $x \rightarrow 1_x$ is continuous on X to N.

2.4 **Theorem:** Suppose X is a Boolean space, for each x in X, a p-near-ring N_x with zero 0_x and identity 1_x is given. Assume that $N_x \cap N_y = \phi$ for $x \neq y$. Let $\mathbf{N} = \bigcup_{x \in U} N_x$. Define $\Pi : \mathbf{N} \rightarrow X$ by $\Pi(n) = x$ if $n \in N_x$. Assume that a topology is imposed on N such that the following axioms are satisfied.

(I). If $n \in \mathbf{N}$, $\exists$ open sets U in **N** with $n \in U$ and $M \subseteq X$ such that $\Pi : U \rightarrow M$ is a homeomorphism.

(II). Let $\mathbf{N} + \mathbf{N} = \{ (m, n) / \Pi(m) = \Pi(n)\}$ with the topology induced by the product topology in $N \times N$. Then the mapping $n \rightarrow -n$ is continuous on **N** to **N**. And the mappings $(m, n) \rightarrow m + n$, $(m, n) \rightarrow m \cdot n$ are continuous on **N + N** to **N**.

(III). The mapping $x \rightarrow 1_x$ is continuous on X to N. Then N is a sheaf of p-near-rings over X

Proof: Suppose U is an open set in X.

Define $N(U) = \{ \sigma / \sigma : U \rightarrow N$ is continuous & $\Pi \circ \sigma = id_u\}$.

Claim: **N** is a presheaf of sets over X. Define for each pair of open sets $V \subseteq U$ of X,

$\rho^{u}{}_{v} : N(U) \rightarrow N(V)$ by $\rho^{u}{}_{v}(\sigma) = \sigma / V$.

(i). Clearly $\rho_u{}^u = id_u$.

(ii). Suppose $W \subseteq V \subseteq U$ (all open) then $\rho^{u}{}_{w} = \rho^{v}{}_{w} \circ \rho^{u}{}_{v}$.

$\therefore$ N is a presheaves of sets.

For every open set U of X, Define +, $\cdot$ on N(U) : For all $x \in U$

$(\sigma_1 + \sigma_2)(x) = \sigma_1(x) + \sigma_2(x)$, $(\sigma_1\sigma_2)(x) = \sigma_1(x)\,\sigma_2(x)$, $0(x) = 0_x$, and $1(x) = 1_x$. with these definitions N(U) is a p-ring with identity.

Claim: N is a monopresheaf.

Suppose U is an open set of X and $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ is an open covering of U,

each U_λ is an open in X, and σ , $\beta \in N(U)$ such that $\rho_{u_\lambda}^{u}(\sigma) = \rho_{u_\lambda}^{u}(\beta)$

$\Rightarrow \sigma / U_\lambda = \beta / U_\lambda \ \forall \lambda \in \Lambda$.

$\therefore \ \sigma = \beta$ on U.

Claim: N satisfies Glueing condition.

Suppose U is clopen in X and $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ is an open covering for U,

U_λ is clopen set in X and $\sigma_\lambda \in N(U_\lambda)$, $\lambda \in \Lambda$, such that

$$\rho^{u_\lambda}_{u_\lambda \cap u_u} (\sigma_\lambda) = \rho^{u_u}_{u_\lambda \cap u_u} (\sigma_u),$$

$\Rightarrow \quad \sigma_\lambda / u_\lambda \cap u_u = \sigma_\mu / u_\lambda \cap u_u$

Since $X = U \cup U^C$

$= \bigcup_{\lambda\in\Lambda} U_\lambda \cup U^C$. X is Boolean space so has partition property,

$\exists$ a finite set $\{M_1, M_2, \ldots\ldots\ldots, M_r\}$, of clopen sets in X, such that

$M_i \cap M_j = \phi$ for $i \neq j$, $M_i \subseteq N_{\lambda i}$ and $\bigcup_{i=1}^{r} M_i = U$

Let $\sigma = \{\sigma_{\lambda 1} \cup \sigma_{\lambda 2} \cup, \ldots\ldots\ldots, \cup \sigma_{\lambda r}\} \Rightarrow \sigma \in N(U)$.

And $\sigma / U_\lambda = \sigma_\lambda$ i.e $\rho_{u_\lambda}^{u}(\sigma) = \sigma_\lambda$

$\therefore$ Glueing condition is satisfied for clopen sets.

Since X is Boolean space, so X has clopen base, the Glueing condition is satisfied for every open set.

$\therefore$ N is a Sheaf of p-near-rings over X.

2.5 **Theorem[10]:** Suppose N is a Sheaf of p-near-rings with identity over a Boolean space X. Let Y be a closed subset of X, and suppose that $\sigma \in N(Y)$. Then $\exists \ \tau \in N(X) \ni \tau / Y = \sigma$.

2.6 **Note:** Let Σ be a sheaf of p-near-rings over X. Let U be a subset of X. a section of Σ over U is a continuous mapping σ of U to Σ such that $\Pi(\sigma(x)) = x$ for all $x \in U$. The class of all sections of Σ over U is denoted by $\Sigma(U)$.

2.7 **Theorem:** Every p-near-ring with identity is isomorphic with sheaf of p-rings with identity : Suppose N is a p-near-ring with identity.

The disjoint union

$\mathbf{N} = \bigcup_{x \in X} N / Nx$ is a sheaf with basic open set $\hat{n}(\Gamma(e))$ for all e in B(N) and $n \in N$, where $\hat{n} : X \to \Sigma$, where X = Spec B(N), $\hat{n}(p) = \bar{n}$ in N/p.

Proof: Define $\prod : \mathbf{N} \to X$ by $\prod(x) = x$ if $n \in N/Nx$.

We are going to impose a topology on Σ by taking

For x in $\Gamma(e^{l}) \cap \Gamma(e^{ll})$,

Let $\hat{n}_1(x) = \hat{n}_2(x)$ with n_1, n_2 in N $\Rightarrow \bar{n}_1 = \bar{n}_2$ in N/ Nx

$\Rightarrow$ $n_1 - n_2 = e_{n1-n2}\, s$ for some s in N e_{n1-n2} in Nx

$\therefore$ $\bar{n}_1 = \bar{n}_2$ in N/ Nx for all x in $\Gamma(1 - e_{n1-n2})$

Denote $1 - e_{n1-n2}$ by e.

$\therefore$ $\bar{n}_1 = \bar{n}_2$ in N/ Nx for all x in $\Gamma(e^{l}e^{ll}e)$.

$\therefore$ $\hat{n}_1(\Gamma(e^{l}e^{ll}e)) \subseteq \hat{n}_1(\Gamma(e^{l})) \cap \hat{n}_2(\Gamma(e^{ll}))$

$\therefore$ $\{ \hat{n}(\Gamma(e)) / n$ in N, $e \in B(N)\}$ forms a basis for a topology on N

Let $x \in \hat{n}^{-1}(\hat{n}_1(\Gamma(e))$ for n, n_1 in N and e in B(N)

$\Rightarrow \hat{n}(x) = \hat{n}_1(x)$ for some x in $\Gamma(e)$

$\Rightarrow \hat{n}(x^{l}) = \hat{n}_1(x^{l})$ in some open set. $\hat{n}(\Gamma(e^{l}))$ contained in $n^{l}(\hat{\Gamma}(e^{l}))$. Then $\hat{n}$ is a section for all in N.

Then we can easily verify the following conditions.

(i) If $n \in \mathbf{N}$, $\exists$ open sets U in $\mathbf{N}$ with $n \in U$ and $M \subseteq X \ni \Pi : U \to M$ is a homeomorphism.

(ii) Let $\mathbf{N} + \mathbf{N} = \{ (m, n) / \Pi(m) = \Pi(n) \}$ with the topology by the product topology in $\mathbf{N} \times \mathbf{N}$. Then the mapping $n \to -n$ is continuous and mapping $(m, n) \to m + n$, $(m, m) \to mn$ are continuous on $\mathbf{N} + \mathbf{N}$ to $\mathbf{N}$.

(iii)The mapping $x \to 1_x$ is continuous on X to Σ

Clearly (i) $\Gamma(X)$ the set of all global sections is a p-near-ring by point wise operations i.e.

$(\sigma_1+\sigma_2)(x) = \sigma_1(x) + \sigma_2(x)$, $(\sigma_1\sigma_2)(x) = \sigma_1(x)\,\sigma_2(x)$, $0(x) = 0_x$, and $1(x) = 1_x$.

(2) N is isomorphic with p-near-ring of global sections.

(3) The near-ring with identity is isomorphic with p-near-ring of global sections $\Gamma(X)$ of the sheaf $N = \bigcup_{x\in X} N/Nx$.

(4) Since X is a Boolean space, the ringed space (X, N) is reduced.

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