QUASI -STATIC THERMAL STRESSES OF A THICK CIRCULAR PLATE SUBJECTED TO RANDOM TEMPERATURE DISTRIBUTION ON ITS UPPER FFACE

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***Abstract:** As we know, thermal behavior of structure must be considered in many situation, such as study of thermal stresses, strains, displacement. There is practical requirement of thick circular plate in various modern project. Circular plate is one of the most widely used in various engineering application such as pavement of highways and airports, building walls and bridge decks. In most cases, the plate have to carry various loads. Therefore understanding of the mechanics of the plate is essential for designer. In this task I endeavour to solve the differential equation of heat conduction, by applying Integral transform and supplying transient heat flux externally along the axial direction to thick circular plate of radius a and thickness h, which is free from traction. The initial temperature of the plate is same as that of the surrounding medium, which is zero. The result is obtained in series form of Bessel's function. The result is illustrated numerically and graphically. The obtained result may be useful in solving engineering problem, particularly for industrial problem, machines subjected to heating and cooling.*

Keywords: Transient heat conduction, Temperature, Stresses, Displacement potential.

1. INTRODUCTION

Circular plate is one of the most widely used in various engineering application such as pavement of highways and airports, building walls and bridge decks. In most cases, the plate have to carry various loads. Therefore understanding of the mechanics of the plate is essential for designer. The problem of the thermoelasticity in thin circular plate were considered by W. Nowacki (1957)and Roy Choudhuri (1973), Wankhede (1982) was determined the quasi-static thermal stresses in a thin circular plate. V. S. Kulkarni, K. C. Deshmukh (2008) and K. R. Gaikwad, K. P. Ghadle(2012) have determined thermal stresses in thick circular plate by using separation of variable method and G. D. Khedar, K. C. Deshmukh (2011) were determined thermal stresses in thick circular plate .It is believed that this problem has not been considered .

2. FORMULATION OF A HEAT CONDUCTION PROBLEM

Consider a circular thick plate of thickness h occupies space $0 \le r \le a$ $-h/2 \le z \le h/2$

The temperature of the plate at time t satisfies differential equation of heat conduction

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{k}\frac{\partial T}{\partial t} \tag{1}$$

Where k is thermal diffusivity of the material of the plate. The initial and boundary condition are

$$T = 0 \quad \text{at} \quad t=0 \tag{2}$$

$$\frac{\partial T}{\partial r}=0 \quad \text{at} \quad r=a \tag{3}$$

$$T = 0 \quad \text{at} \quad z=-\frac{h}{2} \tag{4}$$

$$T=\psi(t)\phi(r) \quad \text{at} \quad z=\frac{h}{2} \tag{5}$$

Equations (1) to (5) constitute a mathematical formulation of the problem.

3. SOLUTION OF HEAT CONDUCTION PROBLEM:

We define finite Hankel transform of $T(r,z,t)$ by

$$T_H=T(\xi_n,z,t)=\int_0^a T(r,z,t)rJ_0(\xi_n r)dr \tag{6}$$

And it's inverse by

$$T(r,z,t)=\frac{2}{a^2}\sum_{n=0}^{\infty}T(\xi_n,z,t)\frac{J_0(\xi_n r)}{J_1^2(\xi_n a)} \tag{7}$$

Where ξ_n is root of the transcendental equation

$$J_1(\xi_n a)=0 \tag{8}$$

Taking Laplace Transform of equation (1) to (5)

we get $\frac{\partial^2 \overline{T}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{T}}{\partial r}+\frac{\partial^2 \overline{T}}{\partial z^2}=\frac{s}{k}\overline{T}$ (9)

$$\overline{T}=0 \quad \text{at} \quad t=0 \tag{10}$$

$$\frac{\partial \overline{T}}{\partial r}=0 \quad \text{at} \quad r=a \tag{11}$$

$$\overline{T}=0 \quad \text{at} \quad z=-\frac{h}{2} \tag{12}$$

$$\overline{T}=\overline{\psi}(s)\phi(r) \quad \text{at} \quad z=\frac{h}{2} \quad 0<r<r_0,\ r_0<a \tag{13}$$

Where s is parameter of Laplace transform and $\overline{T}$ is Laplace transform of $T(r,z,t)$

By taking finite Hankel transform of equation (9) we obtain

$$\frac{d^2\overline{T}_H}{dz^2} - \xi n^2 \overline{T}_H = \frac{s}{k}\overline{T}_H$$

By solving this equation and applying condition (12) we obtain

$$\overline{T}_H = A_n Sinhp\left(z+\frac{h}{2}\right)$$

By taking inverse Hankel transform we obtain

$$\overline{T} = \frac{2}{a^2}\sum_{n=0}^{\infty} A_n sinhp\left(z+\frac{h}{2}\right)\frac{J_0(\xi_n r)}{J_0^2(\xi_n a)} \tag{14}$$

$$\frac{\partial \overline{T}}{\partial r} = \frac{2}{a^2}\sum_{n=0}^{\infty} A_n sinhp\left(z+\frac{h}{2}\right)\frac{\xi_n J_0'(\xi_n r)}{J_0^2(\xi_n a)}$$

$$\frac{\partial \overline{T}}{\partial r} = -\frac{2}{a^2}\sum_{n=0}^{\infty} A_n sinhp\left(z+\frac{h}{2}\right)\frac{\xi_n J_1(\xi_n r)}{J_0^2(\xi_n a)}$$

By applying condition (11) we obtain $J_1(\xi_n a)=0$ this is transcendental equation

We assume that $\phi(r) = \sum_{n=0}^{\infty} B_n J_0(\xi_n r)$

By multiplying by $rJ_0(\xi_n r)$ and integrating between limit 0 to a we obtain

$$\int_0^a \phi(r) r J_0(\xi_n r)dr = B_n \int_0^a r J_0^2(\xi_n r)dr$$

$$B_n = \frac{2r_0}{a^2\xi_n}\frac{J_1(\xi_n r_0)}{J_0^2(\xi_n a)} \qquad \phi(r) = \sum_{n=0}^{\infty}\frac{2r_0}{a^2\xi_n}\frac{J_1(\xi_n r_0)}{J_0^2(\xi_n a)}J_0(\xi_n r)$$

From (13) and (14) we obtain $A_n = \dfrac{\overline{\psi}(s) r_0 J_1(\xi_n r_0}{\xi n\, sinh(ph)}$

$$\overline{T} = \frac{2r_0}{a^2}\sum_{n=0}^{\infty}\frac{\overline{\psi}(s)J_1(\xi_n r_0)J_0(\xi_n r)}{\xi_n J_0^2(\xi_n a)}\frac{sinh(z+\frac{h}{2})}{sin(ph)} \tag{15}$$

By taking inverse Laplace transform we obtain

$$T(r,z,t) = D\sum_{n=0}^{\infty}(-1)^{n+1} n\frac{J_1(\xi_n r_0)J_0(\xi_n r)}{\xi_n J_0^2(\xi_n a)} sin\alpha_n\left(z+\frac{h}{2}\right)f_n(t) \tag{16}$$

Where $\alpha_n = \frac{n\pi}{h}$ $\beta_n = \xi_n^2 + \frac{n^2\pi^2}{h^2}$ $D = \frac{4kr_0\pi}{a^2h^2}$ $f_n(t) = \int_0^t \psi(t')e^{-k\beta_n(t-t')}dt'$

Temperature change $\tau = T - T_i = T - 0 = T$ where T_i is initial temperature

$$\tau = D\sum_{n=0}^{\infty}(-1)^{n+1} n \frac{J_1(\xi_n r_0) J_0(\xi_n r)}{\xi_n J_0^2(\xi_n a)} \sin\alpha_n \left(z+\frac{h}{2}\right) f_n(t) \tag{17}$$

4. THERMOELASTIC PROBLEM

The differential equation governing the displacement potential function $\varphi(r,z,t)$ as in [6] is given by

$$\frac{\partial^2 \varphi}{\partial r^2}+\frac{1}{r}\frac{\partial \varphi}{\partial r}+\frac{\partial 2\varphi}{\partial z2} = m\tau \quad \text{with} \quad \varphi=0 \quad \text{at } t=0 \tag{18}$$

Where m is the restraint coefficient

The displacement function in the cylinder coordinate system is represented by displacement potential and Michell's function as

$$u_r = \frac{\partial \varphi}{\partial r} - \frac{\partial 2M}{\partial r \partial z} \tag{19}$$

$$u_z = \frac{\partial \varphi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \tag{20}$$

Michell's function M must satisfy $\nabla^2\nabla^2 M = 0$ (21)

$$\text{Where} \quad \nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \tag{22}$$

The component of the stresses are represented by the thermoelastic displacement potential φ and Michell's function M as

$$\sigma_{rr} = 2G\left\{\frac{\partial^2\varphi}{\partial r^2} - m\tau + \frac{\partial}{\partial z}\left[\nu\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right]\right\} \tag{23}$$

$$\sigma_{\theta\theta} = 2G\left\{\frac{1}{r}\frac{\partial \varphi}{\partial r} - m\tau + \frac{\partial}{\partial z}\left[\nu\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right]\right\} \tag{24}$$

$$\sigma_{zz} = 2G\left\{\frac{\partial^2\varphi}{\partial z^2} - m\tau + \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right]\right\} \tag{25}$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial}{\partial z}\left[(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right]\right\} \tag{26}$$

Where G is shear modulus and ν is poisson's ratio

For traction free surface $\sigma_{zz} = \sigma_{rz} = 0$ at $z = \pm \frac{h}{2}$ (27)

5. SOLUTION OF THERMOELASTIC PROBLEM

Displacement potential function $\varphi(r, z, t)$ is given from equation (17) and (18)

$$\varphi = C \sum_{n=0}^{\infty} Y_n J_0(\xi_n r) \sin \alpha_n \left(z + \frac{h}{2} \right) f_n(t) \tag{28}$$

Where $C = \dfrac{4 k m r_0 \pi}{a^2 h^2}$, $Y_n = \dfrac{(-1)^n n J_1(\xi_n r_0)}{\xi_n (\xi_n^2 + \alpha_n^2) J_0^2(\xi_n a)}$

We take suitable form of M satisfying equation (21) as

$$M = C \sum_{n=0}^{\infty} Y_n J_0(\xi_n r) \left(z + \frac{h}{2} \right) \left[A_n \sinh \xi_n \left(z + \frac{h}{2} \right) + B_n \cosh \xi_n \left(z + \frac{h}{2} \right) \right]$$

Where A_n and B_n are constants to be determined from condition (27)

We obtain $A_n = \dfrac{\alpha_n}{2v} f_n(t)$ $\quad B_n = 0$

Therefore $M = C \sum_{n=0}^{\infty} Y_n \dfrac{\alpha_n}{2v} J_0(\xi_n r) \left(z + \dfrac{h}{2} \right) \sinh \left(z + \dfrac{h}{2} \right) f_n(t)$ (29)

Expression for displacement and stresses are

$$u_r = C \sum_{n=0}^{\infty} Y_n J_1(\xi_n r) \left\{ \begin{array}{l} \frac{\alpha_n}{2v} \left[\left(z + \frac{h}{2} \right) \xi_n^2 \cosh \xi_n \left(z + \frac{h}{2} \right) + \xi_n \sinh \xi_n \left(z + \frac{h}{2} \right) \right] - \\ \xi_n \sinh \alpha_n \left(z + \frac{h}{2} \right) \end{array} \right\} f_n(t) \tag{30}$$

$$u_z = C \sum_{n=0}^{\infty} Y_n J_0(\xi_n r) \left\{ \begin{array}{l} \alpha_n \cos \alpha_n \left(z + \frac{h}{2} \right) + \left(\frac{1}{v} - 2 \right) \alpha_n \xi_n \cosh \xi_n \left(z + \frac{h}{2} \right) - \\ \frac{\alpha_n \xi_n^2}{2v} \left(z + \frac{h}{2} \right) \sinh \xi_n \left(z + \frac{h}{2} \right) \end{array} \right\} f_n(t) \tag{31}$$

$$\frac{\sigma_{rr}}{2G} = C \sum_{n=0}^{\infty} Y_n \left\{ \begin{array}{l} \left[\xi_n \frac{J_1(\xi_n r)}{r} + \alpha_n^2 J_0(\xi_n r) \right] \sin \alpha_n \left(z + \frac{h}{2} \right) + \left[\alpha_n \xi_n^2 J_0(\xi_n r) \sinh \xi_n \left(z + \frac{h}{2} \right) \right] - \\ \frac{\alpha_n}{2v} \left[\xi_n \frac{J_1(\xi_n r)}{r} - \xi_n^2 J_0(\xi_n r) \right] \left[\xi_n \left(z + \frac{h}{2} \right) \cosh \xi_n \left(z + \frac{h}{2} \right) + \sinh \xi_n \left(z + \frac{h}{2} \right) \right] \end{array} \right\} f_n(t) \tag{32}$$

$$\frac{\sigma_{\theta\theta}}{2G}=C\sum_{n=0}^{\infty}Y_n\left\{\begin{array}{l}\left[\left(\xi_n^2+\alpha_n^2\right)J_0(\xi_n r)-\xi_n\frac{J_1(\xi_n r)}{r}\right]\sin\alpha_n\left(z+\frac{h}{2}\right)+\left[\alpha_n\xi_n^2 J_0(\xi_n r)\sinh\xi_n\left(z+\frac{h}{2}\right)\right]- \\ \frac{J_1(\xi_n r)}{r}\frac{\alpha_n\xi_n}{2\nu}\left[\left(z+\frac{h}{2}\right)\xi_n\cosh\xi_n\left(z+\frac{h}{2}\right)+\sinh\xi_n\left(z+\frac{h}{2}\right)\right]\end{array}\right\}f_n(t) \tag{33}$$

$$\frac{\sigma_{zz}}{2G}=C\sum_{n=0}^{\infty}Y_n J_0(\xi_n r)\left\{\begin{array}{l}\xi_n^2\sin\alpha_n\left(z+\frac{h}{2}\right)+\left(\frac{1}{2\nu}-1\right)\alpha_n\xi_n^2\sinh\xi_n\left(z+\frac{h}{2}\right)- \\ \left(z+\frac{h}{2}\right)\frac{\alpha_n}{2\nu}\xi_n^3\cosh\xi_n\left(z+\frac{h}{2}\right)\end{array}\right\}f_n(t) \tag{34}$$

$$\frac{\sigma_{rz}}{2G}=C\sum_{n=0}^{\infty}Y_n\left\{\begin{array}{l}\alpha_n\xi_n\cosh\xi_n\left(z+\frac{h}{2}\right)+\frac{\alpha_n\xi_n^2}{2\nu}\left(z+\frac{h}{2}\right)\sinh\xi_n\left(z+\frac{h}{2}\right)- \\ \alpha_n\xi_n\cos\alpha_n\left(z+\frac{h}{2}\right)\end{array}\right\}f_n(t) \tag{35}$$

6.NUMERICAL CALCULATION

Set $\psi(t)=e^t$ $\quad f_n(t)=\dfrac{e^t-e^{-k\beta_n t}}{1+k\beta n}$ $\quad a=1m$ $\quad h=.5m$ $\quad r_0=.5\text{m}$ $\quad 2GC=A$

$k=112.34\times10^{-6}m^2s^{-1}$ $\quad \nu=0.35$ $\quad \alpha_n=\dfrac{n\pi}{h}$ $\quad \beta_n=\xi_n^2+\dfrac{n^2\pi^2}{h^2}$

$$Y_n=\frac{(-1)^n nJ_1(.5\xi_n)}{\xi_n(\xi_n^2+\alpha_n^2)J_0^2(\xi_n)}$$

Above equation yields as

$$\frac{T(r,z,t)}{D}=-\sum_{n=0}^{\infty}Y_n(\xi_n^2+\alpha_n^2)J_0(\xi_n r)sin\alpha_n(z+.5)f_n(t) \tag{36}$$

$$\frac{u_r}{A}=\sum_{n=0}^{\infty}Y_nJ_1(\xi_n r)\left\{\begin{array}{l}\frac{\alpha_n}{2\nu}\left[(z+.25)\xi_n^2\cosh\xi_n(z+.25)+\xi_n\sinh\xi_n(z+.25)\right]- \\ \xi_n\sinh\alpha_n(z+.25)\end{array}\right\}f_n(t) \tag{37}$$

$$\frac{u_z}{A}=\sum_{n=0}^{\infty}Y_nJ_0(\xi_n r)\left\{\begin{array}{l}\alpha_n\cos\alpha_n(z+.25)+\left(\frac{1}{\nu}-2\right)\alpha_n\xi_n\cosh\xi_n(z+.25)- \\ \frac{\alpha_n\xi_n^2}{2\nu}(z+.25)\sinh\xi_n(z+.25)\end{array}\right\}f_n(t) \tag{38}$$

$$\frac{\sigma_{rr}}{A}=\sum_{n=0}^{\infty}Y_n\left\{\begin{array}{l}\left[\xi_n\frac{J_1(\xi_n r)}{r}+\alpha_n^2 J_0(\xi_n r)\right]\sin\alpha_n(z+.25)+\left[\alpha_n\xi_n^2 J_0(\xi_n r)\sinh\xi_n(z+.25)\right]- \\ \frac{\alpha_n}{2\nu}\left[\xi_n\frac{J_1(\xi_n r)}{r}-\xi_n^2 J_0(\xi_n r)\right]\left[\xi_n(z+.25)\cosh\xi_n(z+.25)+\sinh\xi_n(z+.25)\right]\end{array}\right\}f_n(t) \tag{39}$$

$$\frac{\sigma_{\theta\theta}}{A}=\sum_{n=0}^{\infty}Y_n\left\{\begin{array}{l}\left[\left(\xi_n^2+\alpha_n^2\right)J_0(\xi_n r)-\xi_n\frac{J_1(\xi_n r)}{r}\right]\sin\alpha_n(z+.25)+\left[\alpha_n\xi_n^2 J_0(\xi_n r)\sinh\xi_n(z+.25)\right]- \\ \frac{J_1(\xi_n r)}{r}\frac{\alpha_n\xi_n}{2\nu}\left[(z+.25)\xi_n\cosh\xi_n(z+.25)+\sinh\xi_n(z+.25)\right]\end{array}\right\}f_n(t) \tag{40}$$

$$\frac{\sigma_{zz}}{A}=\sum_{n=0}^{\infty}Y_n J_0(\xi_n r)\left\{\begin{array}{l}\xi_n^2\sin\alpha_n(z+.25)+\left(\frac{1}{2\nu}-1\right)\alpha_n\xi_n^2\sinh\xi_n(z+.25)- \\ (z+.25)\frac{\alpha_n}{2\nu}\xi_n^3\cosh\xi_n(z+.25)\end{array}\right\}f_n(t) \tag{41}$$

$$\frac{\sigma_{rz}}{A}=\sum_{n=0}^{\infty}Y_n\left\{\begin{array}{l}\alpha_n\xi_n\cosh\xi_n(z+.25)+\frac{\alpha_n\xi_n^2}{2\nu}(z+.25)\sinh\xi_n(z+.25)- \\ \alpha_n\xi_n\cos\alpha_n(z+.25)\end{array}\right\}f_n(t) \tag{42}$$

Numerical calculation has been carried out for Copper plate and graphs are drawn in Microsoft office Excel 2007

With $\xi_0=3.8317$ $\xi_1=7.0156$ $\xi_2=10.1735$ $\xi_3=13.3237$ $\xi_4=16.470$ $\xi_5=19.6159$ $\xi_6=22.7601$ $\xi_7=25.9037$ $\xi_8=29.0468$ $\xi_9=32.18$ being the positive roots of transcendental equation $J_1(\xi_n a)=0$

7. CONCLUSION AND DISCUSSION

1) From figure No.1 and 2 we observe that temperature increases as z-increases and r-increases with t increases.
2) From figure no.3 and 4 we observe that radial displacement decreases as z-increases and r-increases with t increases.
3) From figure No. 5 and 6 we observe that axial displacement increases as z-increases and r-increases.
4) From figure No. 7 and 8 we observe that radial stress decreases as z- increases and r-increases with time increases.
5) From figure no. 9 and 10 we observe that angular stress decreases as z- increases and r-increases with time increases.
6) From figure No. 11 and 12 we observe that axial stress increases as z-increases and r-increases with t increases
7) From figure No. 13 and 14 we observe that shear stress decreases as z-increases and r-increases with t increases and it is same for all radius.

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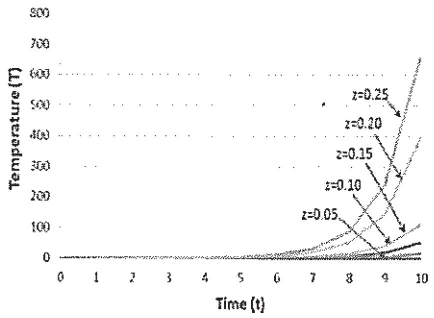

Figure No.1 T verses t for different z

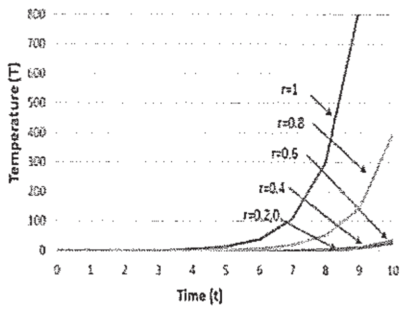

Figure No.2 T verses t for different r

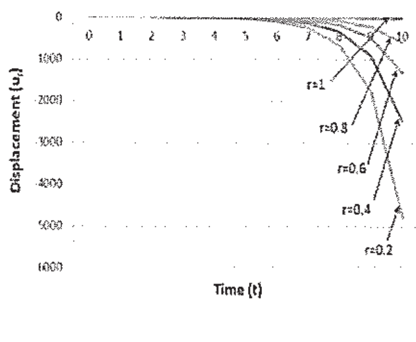

Figure No.3 u_r verses t for different z

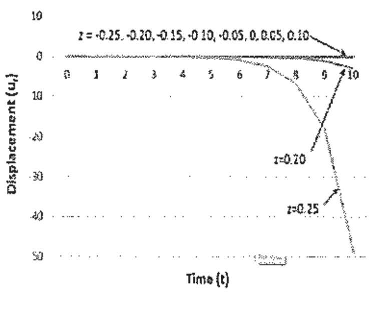

Figure No.4 u_r verses t for different r

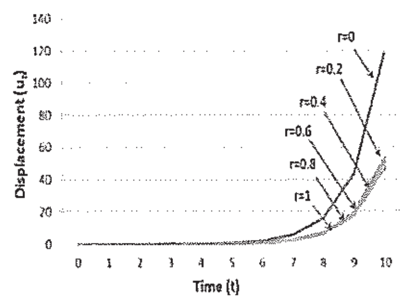

Figure No.5 u_z verses t for different z

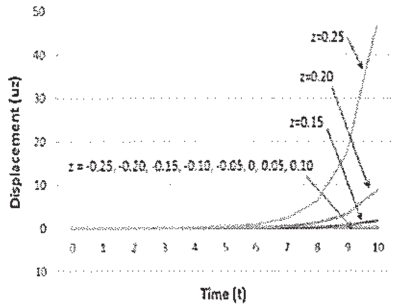

Figure No.6 u_z verses t for different r

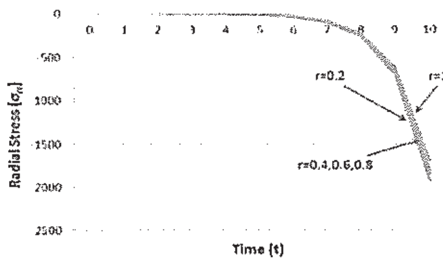

Figure No.7 σ_{rr} verses t for different z

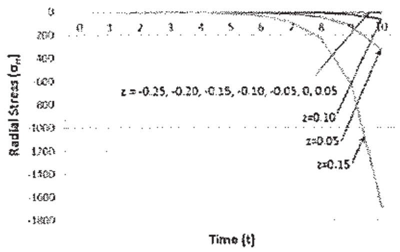

Figure No.8 σ_{rr} verses t for different r

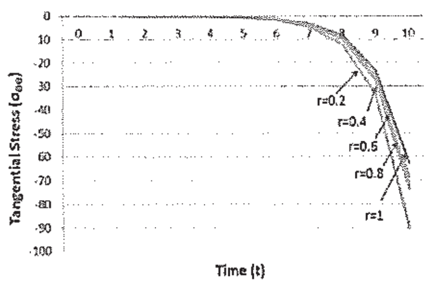

Figure No.9 $\sigma_{\theta\theta}$ verses t for different z

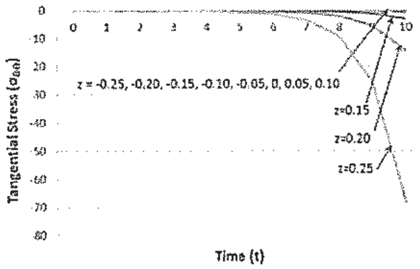

Figure No.10 $\sigma_{\theta\theta}$ verses t for different r

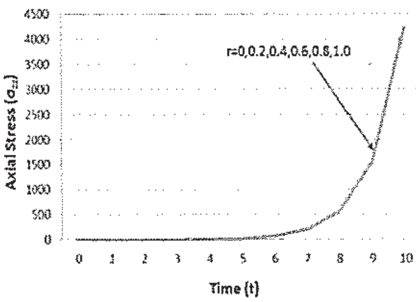

Figure No.11 σ_{zz} verses t for different z

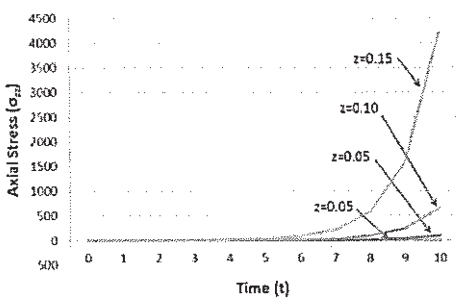

Figure No.12 σ_{zz} verses t for different r

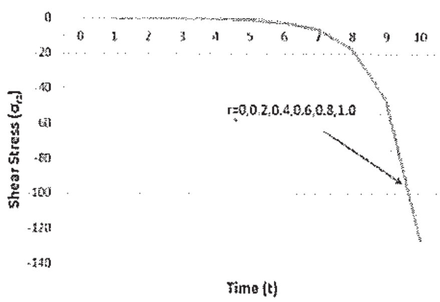

Figure No.13 σ_{rz} *verses t for different z*

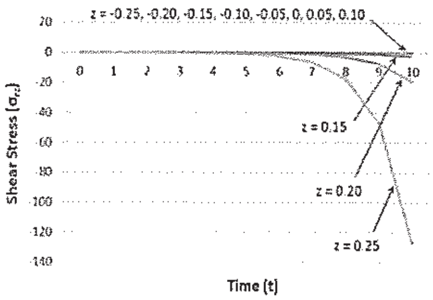

Figure No.14 σ_{rz} *verses t for different r*

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