RIGHT ALTERNATIVE RINGS WITH SQUARES IN THE ALTERNATIVE NUCLEUS

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Abstract: *The associator (x, y, z) is defined by (x, y, z) = (xy) z- x (yz) for all x, y, z in R. The commutator (x,y) is defined by (x,y) = xy-yx for all x,y in R. R is called n-divisible (n, a natural number) if nx = 0 implies x = 0 for all $x \in R$. R is said to be prime if whenever A and B are ideals of R such that AB = 0, then either A = 0 or B = 0.In this paper we prove that if R is a 2-divisible prime right alternative ring with (xy)x = (xx)y for all elements x, y $\in$ R, then R must be alternative.*

1. INTRODUCTION

Pchelincev [2] has given an example of a right nilpotent right alternative ring with $R^2 \subset N$ which is not nilpotent and not alternative. Slin΄KO [3] has given an example of a right alternative ring with $R^2 \subset N$ which is not alternative and left nilpotent but not nilpotent. In [1] Kleinfeld and Smith studied right alternative rings under the assumption that all squares x^2 are in one of the nuclei. In this paper we show that the square of every element x of a right alternative ring R satisfying (xy) x = (xx) y is in the alternative nucleus N_β. Using this we prove that a 2-divisible prime right alternative ring is alternative.

2. PRELIMINARIES

A ring R is defined to be right alternative
if (x, y, y) = 0, (1)
for all x, y in R.The commutator is defined by (x, y) = xy-yx. The associator (x, y, z) = (xy) z- x (yz) for all x, y, z in R. R is called n-divisible (n, a natural number) if nx = 0 implies x = 0 for all $x \in R$. R is said to be prime if whenever A and B are ideals of R such that AB = 0, then either A = 0 or B = 0. Throughout this paper R will denote a 2-divisible prime right alternative ring with (xy) x = (xx) y. 2

The following identities hold in any ring:

(wx, y, z) – (w,xy,z) + (w,x,yz) – w(x,y,z) –(w,x,y)z = 0 (3)

(xy, z) – x (y,z) – (x,z) y = 2 (x,y,z) + (z,x,y) .... (4)

The following identities hold in any right alternative ring:

$(x, y^2, z) = (x, y, yz + zy)$, (5)

(wx, y,z) + (w, x, (y, z)) = w (x, y, z) + (w, y, z) x, ... (6)

(x,yz,y) = (x,z,y) y, ... (7)

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$(x, (y,y, x)) = (xy, y, x) + (y, x, xy) + (x, xy, y),$ .... (8)

$[(x,y), z] + [(y,z), x] + [(z, x), y] = 2(x,y,z) + 2(y, z, x) + 2(z, x,y),$ (9)

$[(a, b, c), y,z] = [a, b, (c, y, z)] + [a, (b, y, z), c] + [(a, y, z), b, c] - (a, b, c)(y,z) + [a,b,c (y,z)] - [a,b, (y,z)] c.$ (10)

The linearization of 1 we have
$xy.z + xz.y = x (yz + zy).$ ...... (11)

If we define $yz + zy = y \circ z$, then the right hand side of 11 is replaced by $x (yoz)$.

The associator form of 11 is $(x, y, z) + (x, z, y) = 0$. The linearization of 7 we have
$(x, y, wz) + (x, w, yz) = (x, w, z) y + (x, y, z) w.$ (12)

In a right alternative ring, we have

$(xy. z) y = x (yz. y).$ ..... (13)

The left nucleus N_L, the right nucleus N_r, the nucleus N, the alternative nucleus N_β and the commutative center U of R are defined as follows:

$N_L = (n \in R / (n, R, R) = 0)$

$N_r = (n \in R / (R, R, n) = 0)$

$N_\beta = \{n \in R / (x, x, n) = 0\}$

$U = \{u \in R / (R, u) = 0\}.$

All four of these nuclei are subrings, 2-and 3-divisible being assumed for U to be a subring. We define the associator ideal A as $A = \sum(R, R, R) + (R, R, R) R$. Using 11, 2, 13and 2 in that order, we have $(yx^2) y = (yx. x) y = (yx .y) x = y (xy. x) = y (x^2y)$.Thus $(y, x^2, y) = 0$.This shows that for all
$x \in R$ we have $x^2 \in N_\beta$ .... (14)

If $N = N_\beta$ then we know that $(N_\beta, R, R) \subset N_\beta$.

Thus $(N, R, R) \subset N$. .... (15)

From 14 we have $x^2 \in N \Rightarrow xy + yx \in N \Rightarrow xoy \in N$, ... (16)

for all $x,y, \in R$. Substituting $y = x^2$ in 16, we get $2xx^2 = 2x^3 \in N$. Then $x^3 \in N$ Linearization yields $yx. x + xy. x + x^2y \in N$. But 1 implies $yx. x = yx^2$ and 16 implies $yox^2 \in N$. Thus $xy. X \in N$. ..... (17)

Linearization of 17 shows $xy. z + zy. x \in N$. We define $a \equiv b$ if and only if $a - b \in N$.

Then xy. z ≡ - zy. x. ..... (18)

Now let n∈N. Then nx. y ≡ - yx . n ≡ n . yx, using 18 and 6. Since NN ⊂ N, we may use 16 so that n . yx ≡ - n . xy. Thus nx. y ≡ - n . xy. Also15 implies nx . y ≡ n . xy, so that 2nx . y ≡ 0. Hence nx. y∈N and n. xy∈N. ..... (19)

Clearly 19 implies nz . xy∈N. Then noz∈N and 19 implies zn . xy∈N. In fact from this it follows that nz . xy∈N, zn . xy∈N, xy . nz∈N, xy . zn∈N, (20)

using 16. Now let p, q, x, y, z∈R. Then pq . x ≡ - xq . p ≡ p . xq using 18and 16Now 19implies N . RR ⊂ N.

Therefore (pq . x) (yz) ≡ - (xq . p) (yz) ≡ (p. xq) (yz), ...21Using this in conjunction with18 and 16. Then 20 implies RN.RR ⊂ N.
Thus (p. xq) (yz) ≡ - (p . qx) (yz). ..... (22)

Combining 21and 22 we have
(pq.x) (yz) ≡- (p.qx) (yz). .... (23)

2. MAIN RESULTS

To prove the main theorem first we prove the following lemmas:

Lemma 1: Let M be the submodule of R generated by all alternators (x, x, y). Then B = M + MR is an ideal of R.

Proof: From 3 and 5 we obtain for a, b, x∈R, a(b, b, x) = (ab, b, x) – (a, b^2, x) + (a, b, bx) = (ab, b, x) – (a, b, bx) – (a, b, xb) + (a, b, bx) = (ab, b, x) – (a, b, xb). Modulo M thus gives with 7 that a (b, b, x) ≡ - (b, ab, x) – (b, xb, a) = - (b, b, x) a – (b, b, a)x. Hence a (b, b, x) = - (b, b, x) a – (b, b, a) x modulo M, ... (24)

For a, b, x∈R. The right side of 24, hence the left one, too, is symmetric in a and x. Applying 24 to 10

We get ((a, a, c), y, z) ≡ [y, z] (a, a, c) ≡ c (a, a, [y, z]) modulo M, (25)

For a, c, y, z∈R. From 24 and 25 we see (M, R, R) ⊂ M + RM ⊂ M + MR. Trivially M + (M, R, R) = M + (R, R, M). This proves that M + RM is a left ideal and M + MR is an ideal. By [4] we have M W_β = 0. Hence (M + RM)W_β ⊂ (R, M, W_β) ⊂ (M, W_β, R) = 0 and (M + MR)I ⊂ (M, R, I) ⊂ (M, I, R) = 0. Thus B is an ideal of R

Lemma 2: If R is a right alternative ring, then B ⊂ J.

Proof: From [4] we have B is an ideal of R may be described as B = M + MR, where M is the additive span of all alternators (x, x, y) of R. We have (x, x, y) = x^2 y – x . xy. Then 16 and 17 imply – x . xy∈N_β = N, while x^2 y∈NR. Thus M ⊂ N + N R. However if we let J = N + NR , then using 16 and 15 we can Prove that J is an ideal of R. Consequently MR ⊂ JR ⊂ J, so that B ⊂ J.

Theorem 1: $T = \{t \in N / tB = 0 = Bt\}$ is an ideal of R, contained in N.

Proof: Let $x, y \in R$, $b \in B$, $t \in T$. Then $xt . b = - xb . t + x(bot)$, using 11. Since B is an ideal, $- xb . t = 0$. Therefore $xt . b = 0$. Next $tx . b = - tb . x + t(xob) = 0$, using 11. Moreover $b . xt = - (b, x, t)$. Since $t \in N = N_\beta$, we have $-(b, x, t) = (x, b, t)$. But expansion shows $(x, b, t) = xb . t - x . bt = 0$. Thus $b . xt = 0$. Hence $b . tx = b(tox) = bt. x + bx . t$, using 11 and $bt . x + bx . t = 0$. Thus $b . tx = 0$. From [4],we have $(x, x, ty) = (x, x, y) t$.But $(x, x, y) t \in M t \subset Bt = 0$, so that $(x, x, ty) = 0$. Thus $ty \in N_\beta$. But 16 implies $toy \in N$. Thus $yt \in N$. This proves T is a two-sided ideal of R and concludes the proof of the theorem.

Lemma 3: If R is prime and not alternative in addition, then the only ideal of R which is contained in N is 0.

Proof: Let I be such an ideal. Let $c \in I$ and $x, y, z \in R$. Then $(x, x, cy) \subset (x, x, I) \subset (x, x, N) = 0$. Since $c \in I \subset N$, $(x, x, cy) = (x, x, y)c$, so that $(x, x, y)c = 0$. Hence $MI = 0$. Let $m \in M$. Then 11 implies $my . c = - mc . y + m(coy) = 0$. Thus $MR . I = 0$. This shows $BI = 0$. Since R is not alternative, $B \neq 0$. Then prime implies $I = 0$. This completes the proof of the lemma.

Corollary 1: If in addition R is prime and not alternative, then $T = 0$.
Proof: This is immediate consequence of lemma 3 and theorem 1.
Lemma 4: If in addition to the other assumptions R is prime and not alternative, then $N_r = 0$.

Proof: Let $x, y, z \in R$, $n \in N_r$. Then 4.2.4 implies $(x, y, nz) + (x, n, yz) = (x, y, z)n + (x, n, z)y$. However $(R,n,R) = 0$, so that $(x, y, nz) = (x, y, z)n$. .....26 .... (26)

From 3 we have $(x, y, zn) - (x, yz, n) + (xy, z, n) = x(y, z, n) + (x, y, z)n$, so that $(x, y, zn) = (x, y, z) n$.Combining this with 26 we obtain $(x,y, nz) = (x, y, z) n = (x, y, zn)$. ..... (27)

Thus 27 Proves $(x,y,[n,z]) = 0$,so that $[N_r,R] \subset N_r$. .... (28)

However $N_r \subset N_\beta$ and 28 implies $nz - zn \in N_r \subset N_\beta$, while $nz + zn \in N_\beta$ by assumption. Thus $2nz \in N_\beta$, implying $nz \in N_\beta$. Therefore $0 = (x, x, nz) = (x, x, z) n$. Consequently $M N_r = 0$. Now $(R, R, N_r) = 0$, and 28 can be used to prove that $J = RN_r + N_r$ is an ideal, since $R N_r. R = R. N_r R = R [N_r, R] + R. RN_r \subset R N_r \subset J$. But now $M. RN_r = M[R, N_r] + M. N_r R = 0$, so we have $M J = 0$. Let $m \in M$, $x \in R$, $q \in J$. Then $(m, q, x) = mq . x - m . qx = - m . qx \in MJ = 0$. But then 4.2.3 yields $(m, x, q) =0$. Since $-m. xq \in MJ = 0$, we have $mx .q = 0$, or $MR. J = 0$. Thus $BJ = 0$.Then $B \neq 0$ and R is prime yield $J = 0$ and therefore $N_r = 0$. This concludes the proof of the lemma.

Theorem 2 : $(R, R, N_\beta^2) \subset T$.

Proof: Let a, b, c, x, y, z∈R and p, q∈N = N_β. As before (a, a, pb) = (a, a, b) p. From 19 we have pq . b∈N so that (a, a, pq . b) = 0. Since pq∈N, it follows that (a, a, b) . pq = 0. Thus M . NN = 0. (29)

Now 11shows (a, a, b,) c . pq + {(a, a, b) . pq}c = (a, a, b){copq}, so that (a, a, b)c . pq = (a, a, b){copq} because of 29. However copq∈N as a consequence of 16. Therefore (a,a,b)c.pq = (a,a,b) {copq} =(a,a,{copq} b). ..... (30)

We have 12 implies (p, qc, b) + (p, bc, q) = (p, c, b) q + (p, c, q) b. Then 15 and NN ⊂N imply (p, c, q)b∈N. But q∈N implies (p, c, q) = - (c, p, q).
Hence (c, p, q) b ∈ N. ......(31)

Now 18 implies {cp . q}b ≡ - bq . cp, whereas – bq . cp∈N, using 20. Thus {cp . q}b∈N. ..... (32)

Combining 31 with 32 it follows that {c . pq}b∈N. ..... (33)
Again 18 and 20 imply
{pq . c} b ≡ - {bc . pq}∈N, so {pq . c} b ∈ N. ..... (34)

Using 30,33 and 34 we conclude that MR . NN = 0. ... (35)

Then 29 and 35 imply B . NN = 0. ..... (36)

Using 31 and (c, p, q) ob∈N,

It follows that b(c, p, q)∈N. ...... (37)

Now 6 yields (bc, p, q) + (b, c, [p , q]) = b(c, p, q) + (b, p, q)c. The right hand side of the last equation lies in N because of 31and 37.Also (bc, p, q) = - (p, bc, q)∈N using 15. There fore (b, c, [p, q])∈N. ...(38)

We have (bc) (poq)∈R^2N ⊂ N using 19. Also b{c(poq)} = b{cp . q + cq . p}, using 11. However b{cp . q + cq . p} ≡ -{cp . q + cq . p}b ≡ bq . cp + bp . cq as a consequence of 16 and 18. But bq . cp + bp . cq∈N, using 20.

Hence (b, c, poq)∈N (39)

Now 38 and 39 combine to give 2(b, c, pq)∈N, sothat (b, c, pq)∈N. We have shown (R, R, N^2) ⊂N. ..... (40)

Since B is an ideal, it follows from 36 that BR . NN = 0. Also (B . NN)R = 0 follows from 36. Thus 11 implies B(NNoR) = 0. (41)

Next (R, B, NN) = RB . NN – R{B . NN} = 0, using 36 and the fact that RB ⊂ B. Since NN ⊂ N we get (B, R, NN) = - (R, B, NN) = 0, thus B(R . NN) = 0. (42)

Now 41 and 42 yield B(NN . R) = 0 = B(R . NN). .... (43)

Using 12 we see that $(R, R, N^2)B \subset -(R, B, N^2)R + (R, R, BN^2) + (R, B, RN^2)$. Using 43 and $RB \subset B$, we see that $(R, B, RN^2) = 0$. Similarly 36 implies $(R, B, N^2) = 0$, as well as $(R, R, BN^2) = 0$. Thus $(R, R, N^2)B = 0$. ..(44)

Again Starting with $(R, B, RN^2) = 0$, since19 yields $RN^2 \subset N$,
we have $(B, R, RN^2) = 0$.By expansion of the last equation and $BR \subset B$ together with 42
we obtain $B\{R(RN^2)\} = 0$. ......(45)

However $B\{R^2N^2\} \subset B\{RN^2\} = 0$, using 42 Combining this with 45. we obtain $B(R, R, N^2) = 0$. ...... (46)

Now 44, 46 and 40 shows that $(R, R, N^2) \subset T$.This completes the proof of the theorem.

Theorem 3: If R is a 2-divisible prime right alternative ring with $(xy)\,x = (xx)\,y$ for all elements $x, y \in R$, then R must be alternative.

Proof: Suppose that R is not alternative. Then theorem 1 and lemma 3 imply $T = 0$.Then theorem 2 yields $(R, R, N^{2)} = 0$, so that $N^2{}_\beta \subset N_r$. But then lemma 4 implies $NN = 0$. Let $q, n \in N$, $x, y \in R$. Using 11 we have $(qy)n = -(qn)y + q\,(yon)$. Since $yon \in N$ and $NN = 0$, we obtain $(qy)n = 0$. Thus $(x^2y)n = 0$. Also $-x \,.\, xy \equiv xy \,.\, x \in N$, using 17, so that $-(x \,.\, xy)n = 0$. Therefore $(x, x, y)n = 0$. But $(x, x, y)n = (x, x, ny)$. Consequently $ny \in N$. Since $noy \in N$, this shows N is an ideal of R. Since $N^2 = 0$, we have $N = 0$. But then $(x, x, y) = 0$, for all $x, y \in R$. This means R is left alternative, hence alternative. This is contrary to assumption. Thus R must have been alternative to begin with. This completes the proof of the theorem.

3. REFERENCES

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