GRAPH AUTOMORPHISMS OF MODULAR LATTICE

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Abstract: This paper deals with the relations between graph Automorphisms and direct factors of a modular lattice these theorems are valid for an even wider class of algebraic systems.

Keywords: modular lattice, graph Automorphisms, direct factor.

MSC: 2000: 06C10

1. INTRODUCTION

In connection with Birkhoff's problem 6 from [2], the following result has been proved in [6] (by using the results of [3] and [8]):

Each lattice dealt with in the present paper is assumed to be of locally finite (i.e., all its bounded chains are finite).

For a lattice L let $G(L)$ be the corresponding unoriented graph.

An Automorphisms of the graph $G(L)$ is called also a graph Automorphisms of the lattice L. The graph isomorphism of lattices is defined analogously.

We denote by C the class of all finite lattices L such that each automorphism of $G(L)$ turns out to be a lattice automorphism.

$(*)$ Let L be a finite modular lattice. Then the following conditions are equivalent:

(i) L belongs to C.

(ii) No direct factor of L having more than one element is self-dual.

The natural question arises whether in $(*)$ the assumption of modularity can be replaced by the assumption that L is semimodular.

We define the notions of an interval of type (C) in L and of a graph Automorphisms of type (C).

Let A be a direct factor of a lattice L and $\phi \neq X \subseteq L$. We say that A is orthogonal to X if for any $x_1, x_2 \in X$, the components ofx_1 and x_2 in the direct factor A are equal.

Let C_1 be the class of all lattices L such that each graph Automorphisms of type (C) of L is a lattice Automorphisms.

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We prove (by applying the results and the methods of [4], [6] and [8]): $(*_1)$ Let L be a semimodular lattice. Then the following conditions are equivalent:

(i) L Belongs to C_1.

(ii) If A is a direct factor of L such that A is self-dual and orthogonal to each interval of type (C) in L, then A is trivial (i.e., card $A = 1$).

2. PRELIMINARIES

It follows, L is a lattice. For the notion of the unoriented graph $G(L)$ of L cf.

If $x, y \in L, x < y$ and if the interval $[x, y]$ of L is a two-element set, then we write $x \prec y \, or \, y \succ x$.

Hence a graph Automorphisms of L is a one-to-one mapping ϕ *of* L onto L such that, whenever $x, y \in L$ and $x \prec y$, then

either $\phi(x) \prec \phi(y)$ or $\phi(y) \prec \phi(x)$,

either $\varphi^{-1}(x) \prec \varphi^{-1}(y)$ *or* $\varphi^{-1}(y) \prec \varphi^{-1}(x)$.

2.1. Definition. Let L_0 be a sublattice of L such that L_0 is isomorphic to the lattice in Fig. 1; then the convex closure $\overline{L_0}$of L_0 in L is said to be an interval of type (C) in L.

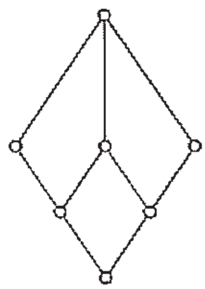

Fig: 1

2.2. Definition. A graph Automorphisms φ of L is said to be of type (C) if, whenever $L_1$is an interval of type (C) in L and $x, y \in L_1$, $x \prec y$, then $\phi(x) \prec \phi(y)$ and $\varphi^{-1}(x) \prec \varphi^{-1}(y)$.

It is easy to verify that if L is modular, then it has no sublattice of type (C); consequently, in this case each graph automorphism of L is of type (C). Therefore $(*)$ is a corollary of $(*_1)$.

We denote by $L^{\sim}$ the lattice dual to L. If L and $L^{\sim}$ are isomorphic, then L is said to be self-dual.

3. AN EXAMPLE

Let us recall that if L can be expressed as a direct product $L_1 \times L_2$ and if $x = (x_1, x_2) \in L$, $y = (y_1, y_2) \in L$, then $x \prec y$ if and only if either $x_1 \prec y_1$ and $x_2 = y_2$, or $x_1 = x_2$ and $y_1 \prec y_2$.

From this we immediately obtain

3.1. Lemma. Let L_1, L_2 be lattices and let φ be a graph isomorphism of L_1 onto L_2. Put $L = L_1 \times L_2$. For each $x = (x_1, x_2) \in L$ we set

$$\varphi(x) = (\varphi^{-1}(x_2), \varphi(x_2)).$$

Then ψ is a graph Automorphism of L.

4. SOME COMMON MISTAKES:

Theorem **4.1**: If $\emptyset \in Hom_R(A, B)$ then $\emptyset A \cong A/\emptyset^{-1}0$. $\emptyset A$ and $\emptyset^{-1}0$ are usually called the image and $Kernal$ of $\emptyset$ respectively.

Theorem **4.2**: Let C be a submodular lattice of A. every submodule of A/C has the form B/C where $C \subset B \subset A$ and $A/B \cong (A/C)(B/C)$.

Theorem **4.3**: If B and C are sub modules of A then $(B + C)/B \cong C/B \cap C$.

*Lemma*1: If $B^1 \subset B \subset A$ *and* $C^1 \subset C \subset A$ then

$$(B^1 + (B \cap C)) / (B^1 + (B \cap C^1)) \cong (C^1 + (B \cap C)) / (C^1 + (B^1 \cap C)).$$

Proof:

We prove that the left side is isomorphic with

$$(B \cap C)/(B^1 \cap C) + (B^1 \cap C)$$

By observing that

$$\left(B^1 + (B \cap C^1)\right) + (B \cap C) = B^1 + (B \cap C)$$

And

$\left(B^1 + (B \cap C^1)\right) \cap (B \cap C) = (B^1 \cap C) + (B^1 \cap C)$,

By symmetry, the Right side is isomorphic to some expression.

By a (finite) chain of modular lattice of A we understand a sequence

$$A_0 \subset A_1 \subset \cdots \subset A_m = A$$

of modular lattice, where each A_i is a modular lattice of A_{i+1/A_i}.

Theorem **4.4**: Given two chains $B = A_0 \subset A_1 \subset \cdots \subset A_m = A$

$$B = B_0 \subset B_1 \subset \cdots \subset B_m = A$$

Then both chains can be refined so that the resulting chains have the same length and isomorphic factors (not necessarily in the same order).

Theorem **4.5**: A Modular is Lattice if and only if every sub module is finitely generated.

Theorem **4.6**: Let B be a modular lattice of A_R. then A is Lattice if and only if B and A/B are Lattice.

Proof:

Assume A Lattice. Since Every submodule of B is a submodule of A, B is an Artinian. Since every modular lattice of A/B has the form C/B where $B \subset C \subset A$, A/B is Lattice.

Conversely assume that A/B and B are Lattice. Consider any descending sequence of modular lattice $A_1 \supset A_2 \supset \cdots$consider now the sequence of modular lattice of A/B:

$$(A_1 + B)/B \supset (A_2 + B)/B \supset \cdots$$

As well as the sequence of modular lattice of B:

$$A_1 \cap B \supset A_2 \cap B \supset \cdots$$

By assumption, both these sequences are ultimately stationary, say after n steps. Then

$A_n \cap B = A_{n+1} \cap B = \cdots$ And

$$(A_n \cap B) / B = (A_{n+1} \cap B) / B = \cdots$$

Hence

$$A_n + B = A_{n+1} + B = \cdots$$

Using the modular law, we now compute

$A_n = A_n \cap (A_n + B) = A_n \cap (A_{n+1} + B) = A_{n+1} \cap (A_n \cap B) = A_{n+1} \cap (A_{n+1} \cap B) = A_{n+1}$.

Corollary: A finite direct product of modular is Lattice if and only if each factor is Lattice.

Theorem **4.7**: A Module has a composition series if and only if it is Lattice.

Theorem **4.8**: An endomorphism of an Lattice module is an automorphism if and *only if it is mono (epimorphism).*

Theorem **4.9**: If f is an endomorphism of the Lattice modular A then, for some n, $A = f^n A + f^{-n} 0$ as a direct sum.

Corollary: If A is indecomposable, Lattice, then every endomorphism of A is either automorphism.

5. REFERENCES

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